Agreeing to Disagree with Limit Knowledge

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Abstract. The possibility for agents to agree to disagree is considered in an extended epistemic-topological framework. In such an enriched context, Aumann's impossibility theorem is shown to no longer hold. More precisely, agents with a common prior belief satisfying limit knowledge instead of common knowledge of their posterior beliefs may actually entertain distinct posterior beliefs. Hence, agents can actually agree to disagree. In particular, agreeing to disagree with limit knowledge is illustrated within a representative epistemic-topological situation.

Keywords: agreeing to disagree, agreement theorems, limit knowledge, interactive epistemology

1 Introduction

The so-called agreement theorem by Aumann [1] establishes the impossibility for two Bayesian agents with a common prior belief to entertain common knowledge of posterior beliefs that are distinct. Understanding two individuals as like-minded if they are both Bayesian and equipped with exactly the same prior information, Aumann's seminal result states that two like-minded individuals that get access to differing information cannot entertain opposing opinions in the case of their opinions being common knowledge. In other words, the agents cannot agree to disagree.

Along these lines Milgrom and Stokey [11] establish an impossibility theorem of speculative trade. Intuitively, their result states that if two traders agree on a prior efficient allocation of goods, then upon receiving private information it cannot be common knowledge that both traders have an incentive to trade. From an empirical or quasi-empirical point of view, the agreement theorem seems quite startling since real world agents do frequently disagree on a

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large variety of issues. It is then natural to scrutinize whether Aumann's basic result still holds with weakened or slightly modified assumptions. In this spirit, Geanakoplos and Polemarchakis [10] drop the assumption of common knowledge of posteriors and show that two Bayesian agents with common priors and finite information partitions already agree on their posterior beliefs after finitely many rounds of communicating their respectively updated posteriors back and forth. Note that, although common knowledge of the posteriors is not needed prior to the communication procedure, it does actually hold after sameness of the posteriors has been established. Moreover, Monderer and Samet [12] replace common knowledge by the weaker concept of common p-belief and establish an agreement theorem with such an approximation of common knowledge. Indeed, they show that if the posteriors of Bayesian agents equipped with a common prior are common p-belief for large enough p, then these posteriors cannot differ significantly. Besides, Samet [13] drops the implicit negative introspection assumption – which states that agents know what they do not know – and shows that Aumann's agreement theorem remains valid with agents ignorant of their own ignorance. Further works on Aumann's agreement theorem are surveyed in Bonanno and Nehring [9].

Here, Aumann's result is revisited in an extended epistemic-topological framework. The epistemic operator common knowledge is replaced by the epistemic-topological operator limit knowledge introduced and studied by Bach and Cabessa [7, 8]. Assuming common priors, Bayesian agents, and limit knowledge of posteriors, we show that the agents' posteriors may differ. Thus, agents can indeed agree to disagree.

2 An Epistemic-Topological Approach to Agreeing to Disagree

Set-based interactive epistemology provides the formal framework in which the agreement theorem is modelled. Having been introduced and notably developed by Aumann [1–6], the discipline provides tools to formalize epistemic notions in interactive situations.

A so-called Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p)$ consists of a countable set Ω of possible worlds, which are complete descriptions of the way the world might be, a finite set of agents I, a possibility partition $\mathcal{I}_i$ of Ω for each agent $i \in I$ representing his information, and a common prior belief function $p : \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$. The cell of $\mathcal{I}_i$ containing the world ω is denoted by $\mathcal{I}_i(\omega)$ and consists of all worlds considered possible by i at world ω. In other words, agent i cannot distinguish between any two worlds ω and ω' that are in the same cell of his partition $\mathcal{I}_i$. Moreover, an event $E \subseteq \Omega$ is defined as a set of possible worlds. For example, the event of it raining in London contains all worlds in which it does rain in London. Note that the common prior belief function p can naturally be extended to a common prior belief measure on the event space $p : \mathcal{P}(\Omega) \to [0, 1]$ by setting $p(E) = \sum_{\omega \in E} p(\omega)$. In this context, it is supposed that each information set of each agent has non-zero prior probability, i.e. $p(\mathcal{I}_i(\omega)) > 0$ for all $i \in I$ and $\omega \in \Omega$. Moreover, all agents are assumed to be Bayesians and to hence update the common prior belief given their private information according to Bayes's rule. More precisely, given some event E and some world ω, the posterior belief of agent i in E at ω is given by $p(E \mid \mathcal{I}_i(\omega)) = \frac{p(E \cap \mathcal{I}_i(\omega))}{p(\mathcal{I}_i(\omega))}$. Farther, an Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I})$ is called finite if Ω is finite and infinite otherwise.

In Aumann's epistemic framework, knowledge is formalized in terms of events. More precisely, the event of agent i knowing E, denoted by $K_i(E)$, is defined as $K_i(E) := \{\omega \in \Omega : \mathcal{I}_i(\omega) \subseteq E\}$. If $\omega \in K_i(E)$, then i is said to know E at world ω. Intuitively, i knows some event E if in all worlds he considers possible E holds. Naturally, the event $K(E) = \bigcap_{i \in I} K_i(E)$ then denotes mutual knowledge of E among the set I of agents. Letting $K^0(E) := E$, m-order mutual knowledge of the event E among the set I of agents is inductively defined by $K^{m+1}(E) := K(K^m(E))$ for all $m \geq 0$. Accordingly, mutual knowledge can also be denoted as 1-order mutual knowledge. Furthermore, an event is said to be common knowledge among a set I of agents whenever all m-order mutual knowledge of it simultaneously hold. Formally, it is standard to define the event that E is common knowledge among the set I of agents as $CK(E) := \bigcap_{m>0} K^m(E)$.

Aumann's agreement theorem states that if two agents entertain a common prior belief function and their posterior beliefs in some event are common knowledge, then these posterior beliefs must coincide. In other words, if two agents with common prior beliefs hold distinct posterior beliefs, then these posterior beliefs cannot be common knowledge among them. Intuitively, it is impossible for agents to consent to distinct beliefs. Thus, agents cannot agree to disagree.

Now, the impossibility for agents to agree to disagree is considered from a topologically enriched epistemic perspective.

In fact, the standard set-based approach to interactive epistemology lacks a general framework providing some formal notion of closeness between events. An amended topological dimension could be capable of introducing an agent perception of closeness of events. In such a more general epistemic-topological framework, the reasoning of agents may thus also depend on topological instead of mere epistemic features of the underlying interactive situation.

In this context, Bach and Cabessa [7, 8] consider Aumann structures equipped with topologies on the event space and introduce the operator limit knowledge, which is linked to epistemic features as well as topological aspects of the event space. More precisely, limit knowledge is defined as the topological limit of higher-order mutual knowledge.

Definition 1. *Let $(\Omega, (\mathcal{I}_i)_{i \in I}, p)$ be an Aumann structure, $\mathcal{T}$ a topology on $\mathcal{P}(\Omega)$, and E an event. If the limit point of the sequence $(K^m(E))_{m>0}$ is unique, then $LK(E) := \lim_{m \to \infty} K^m(E)$ is the event that E is* limit knowledge *among the set I of agents.*

Accordingly, limit knowledge of an event E is constituted by – whenever unique – the limit point of the sequence of iterated mutual knowledge, and thus linked to both epistemic as well as topological aspects of the event space.

Limit knowledge can be understood as the event which is approached by the sequence of iterated mutual knowledge, according to some notion of closeness between events furnished by a topology on the event space. Thus, the higher the iterated mutual knowledge, the closer this latter epistemic event is to limit knowledge.

Note that limit knowledge should not be amalgamated with common knowledge. Indeed, both operators can be perceived as sharing distinct implicative properties with regards to highest iterated mutual knowledge claims. While common knowledge bears a standard implicative relation in terms of set inclusion to highest iterated mutual knowledge, limit knowledge entertains an implicative relation in terms of set proximity with highest iterated mutual knowledge. Besides, limit knowledge also differs from Monderer and Samet's [12] notion of common p-belief. Indeed, common p-belief – as an approximation of common knowledge in the sense of common almost-knowledge – is implied by common knowledge, whereas limit knowledge is not.

Actually, it is possible to link limit knowledge to topological reasoning patterns of agents based on closeness of events. Indeed, agents satisfying limit knowledge of some event are in a limit situation arbitrarily close to entertaining all highest iterated mutual knowledge of this event, and the agents' reasoning may be influenced accordingly. Note that a reasoning pattern associated with limit knowledge depends on the particular topology on the event space, which fixes the closeness relation between events.

The operator limit knowledge is shown by Bach and Cabessa [7, 8] to be able to provide relevant epistemic-topological characterizations of solution concepts in games. Despite being based on the same sequence of higher-order mutual knowledge claims, the distinguished interest of limit knowledge resides in its capacity to potentially differ from the purely epistemic operator common knowledge. Notably, it can be proven that such differing situations necessarily require an infinite event space as well as sequences of higher-order mutual knowledge that are strictly shrinking.[3]

In fact, the topologically amended epistemic framework enables agents with a common prior belief to agree to disagree on their posterior beliefs.

Theorem 1. *There exist an Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p)$ equipped with a topology $\mathcal{T}$ on the event space $\mathcal{P}(\Omega)$, an event $E \subseteq \Omega$, and worlds $\omega, \hat{\omega} \in \Omega$ such that $\omega \in LK(\bigcap_{i \in I}\{\omega' \in \Omega : p(E \mid \mathcal{I}_i(\omega')) = p(E \mid \mathcal{I}_i(\hat{\omega}))\})$, as well as both $p(E \mid \mathcal{I}_i(\hat{\omega})) \neq p(E \mid \mathcal{I}_j(\hat{\omega}))$ and $p(E \mid \mathcal{I}_i(\omega)) \neq p(E \mid \mathcal{I}_j(\omega))$ for some agents $i, j \in I$.*

Proof. Consider the Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p)$, where $\Omega = \{\omega_n : n \geq 0\}$, $I = \{Alice, Bob\}$, $\mathcal{I}_{Alice} = \{\{\omega_{2n}, \omega_{2n+1}\} : n \geq 0\}$, $\mathcal{I}_{Bob} = \{\{\omega_0\}\} \cup \{\{\omega_{2n+1}, \omega_{2n+2}\} : n \geq 0\}$, and $p : \Omega \to \mathbb{R}$ is given by $p(\omega_n) = \frac{1}{2^{n+1}}$ for all $n \geq 0$. Note that the common prior belief function p is well defined since $\sum_{n \geq 0} \frac{1}{2^{n+1}} = 1$. Now, consider the event $E = \{\omega_{2n} : n \geq 1\}$, and the world $\omega_2 \in \Omega$. Besides,

[3] Given some event E, the sequence of higher-order mutual knowledge $(K^m(E))_{m>0}$ is called *strictly shrinking* if $K^{m+1}(E) \subsetneq K^m(E)$ for all $m \geq 0$.

for sake of notational convenience, let the event $\bigcap_{i\in I}\{\omega' \in \Omega : p(E \mid \mathcal{I}_i(\omega')) = p(E \mid \mathcal{I}_i(\omega_2))\}$ be denoted by E'. First of all, observe that $p(E \mid \mathcal{I}_{Alice}(\omega_2)) = \frac{2}{3}$ and $p(E \mid \mathcal{I}_{Bob}(\omega_2)) = \frac{1}{3}$. Moreover, $\{\omega' \in \Omega : p(E \mid \mathcal{I}_{Alice}(\omega')) = p(E \mid \mathcal{I}_{Alice}(\omega_2)) = \frac{2}{3}\} = \Omega \setminus \{\omega_0, \omega_1\}$ and $\{\omega' \in \Omega : p(E \mid \mathcal{I}_{Bob}(\omega')) = p(E \mid \mathcal{I}_{Bob}(\omega_2)) = \frac{1}{3}\} = \Omega \setminus \{\omega_0\}$, whence $E' = (\Omega \setminus \{\omega_0, \omega_1\}) \cap (\Omega \setminus \{\omega_0\}) = \Omega \setminus \{\omega_0, \omega_1\}$. Farther, the definitions of the possibility partitions of *Alice* and *Bob* ensure that $K^m(E') = K^m(\Omega \setminus \{\omega_0, \omega_1\}) = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_{m+1}\}$, for all $m > 0$. Consequently, the sequence $(K^m(E'))_{m>0}$ is strictly shrinking and $CK(E') = \{\omega \in \Omega : \bigwedge_{i\in I} \mathcal{I}_i(\omega) \subseteq E'\} = \emptyset$. Now, consider the topology $\mathcal{T}$ on $\mathcal{P}(\Omega)$ defined by $\mathcal{T} = \{O \subseteq \mathcal{P}(\Omega) : \{\omega_0, \omega_1, \omega_2\} \notin O\} \cup \{\mathcal{P}(\Omega)\}$. Then, the only open neighbourhood of the event $\{\omega_0, \omega_1, \omega_2\}$ is $\mathcal{P}(\Omega)$, and all terms of the sequence $(K^m(E'))_{m>0}$ are contained in $\mathcal{P}(\Omega)$. Thus $(K^m(E'))_{m>0}$ converges to $\{\omega_0, \omega_1, \omega_2\}$. Moreover, for every event $F \in \mathcal{P}(\Omega)$ such that $F \neq \{\omega_0, \omega_1, \omega_2\}$, the singleton $\{F\}$ is open, and since $K^{m+1}(E') \subsetneq K^m(E')$ for all $m > 0$, the strictly shrinking sequence $(K^m(E'))_{m>0}$ will never remain in the open neighbourhood $\{F\}$ of F from some index onwards. Hence $(K^m(E'))_{m>0}$ does not converge to any such event F. Therefore the limit point $\{\omega_0, \omega_1, \omega_2\}$ of the strictly shrinking sequence $(K^m(E'))_{m>0}$ is unique, and $LK(E') = \lim_{m\to\infty} K^m(E') = \{\omega_0, \omega_1, \omega_2\}$. Next, consider the world ω_1. Note that $\omega_1 \in LK(E')$. Also, observe that $p(E \mid \mathcal{I}_{Alice}(\omega_2)) = \frac{2}{3} \neq \frac{1}{3} = p(E \mid \mathcal{I}_{Bob}(\omega_2))$ as well as $p(E \mid \mathcal{I}_{Alice}(\omega_1)) = 0 \neq \frac{1}{3} = p(E \mid \mathcal{I}_{Bob}(\omega_1))$. Finally, taking $\omega = \omega_1$ and $\hat{\omega} = \omega_2$ concludes the proof. $\square$

The preceding possibility result counters Aumann's impossibility theorem in the sense of showing that agents actually can agree to disagree. More precisely, agents may hold distinct actual posterior beliefs, while at the same time satisfying limit knowledge of their posteriors. Hence, agents may agree in the sense of satisfying limit knowledge of their posteriors, while at the same time disagree in the sense of actually entertaining different posterior beliefs.

Generally speaking, the mere fact of topologically enriching the event space concurrently with replacing the purely epistemic operator common knowledge by the epistemic-topological operator limit knowledge enables our possibility result. In such an amended perspective, agents can now be seen to have access to a further dimension in their reasoning that remarkably permits them to agree to disagree on their posterior beliefs. In fact, the agents are in a limit situation of entertaining higher-order mutual knowledge of their posteriors, which, in connection with the particular notion of closeness furnished by the topology, leads them to actually possess different posterior beliefs.

3 A Representative Example

The extension of the standard set-based approach to interactive epistemology with a topological dimension has been shown to enable the possibility for agents to agree to disagree on their posterior beliefs. The question then arises whether agents can still agree to disagree in interactive situations furnished with topologies based on epistemic features. A topology describing a specific agents' perception of the event space is now presented and is then shown to enable agreeing to disagree with limit knowledge.

Towards this purpose, suppose an Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p)$ and an event E. Farther, for any world $\omega \in \Omega$, let E'_ω denote the event consisting of all worlds that induce the same agents' posterior beliefs in E as ω, i.e. $E'_\omega = \bigcap_{i \in I}\{\omega' \in \Omega : p(E \mid \mathcal{I}_i(\omega')) = p(E \mid \mathcal{I}_i(\omega))\}$. Note that constancy of the agents' posterior beliefs in E yields an equivalence relation on the set of possible worlds, and hence every E'_ω represents an equivalence class of worlds. Consequently, the collection $\mathcal{C} = \{E'_\omega : \omega \in \Omega\}$ of all equivalence classes of worlds that induce a same posterior belief profile forms a partition of Ω. Given some event E and some index $m^* > 0$, the epistemically-based topology $\mathcal{T}_{E,m^*}$ is defined as the topology on the event space $\mathcal{P}(\Omega)$ generated by the subbase

$$\begin{aligned}
&\{\{K^m(E'_\omega) : m \geq 0\} : \omega \in \Omega\} \\
\cup\ &\{\mathcal{P}(\Omega) \setminus \{K^m(E'_\omega) : m \geq 0 \text{ and } \omega \in \Omega\}\} \\
\cup\ &\{\{K^m(E'_\omega)\} : 0 \leq m < m^* \text{ and } \omega \in \Omega\} \\
\cup\ &\{\{K^{m^*+j}(E'_\omega) : 0 < j \leq n\} : n > 0 \text{ and } \omega \in \Omega\}.
\end{aligned}$$

The topology $\mathcal{T}_{E,m^*}$ is illustrated in Figure 1, where the infinite sequence $(K^m(E'_\omega))_{m \geq 0}$ is represented by a horizontal sequence of points for each $\omega \in \Omega$, and open sets of the subbase by circle-type shapes around these points.

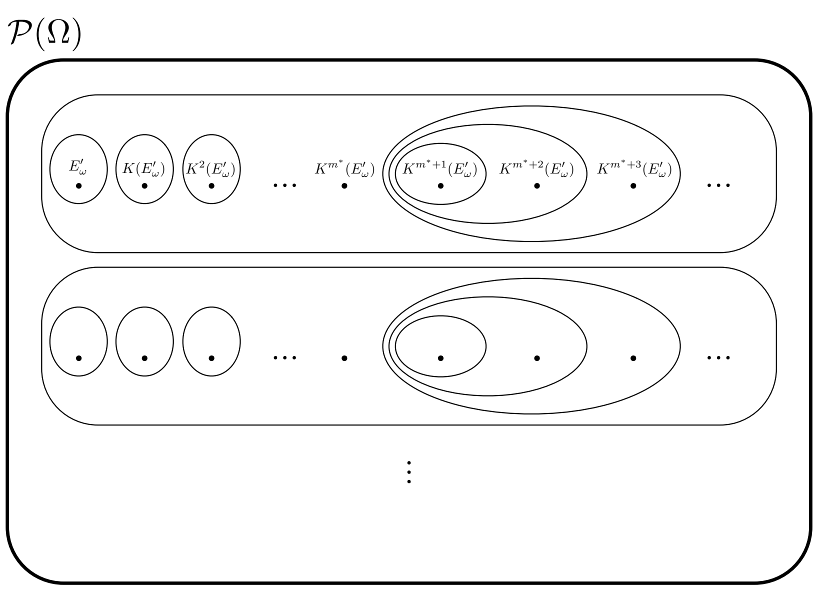

Fig. 1. Illustration of the topology $\mathcal{T}_{E,m^*}$

The topology $\mathcal{T}_{E,m^*}$ reveals a specific agent perception of the event space, according to which the agents entertain a more refined distinction between the m^* first iterated mutual knowledge of their posterior beliefs in E than between the remaining ones. This specific perception is formally reflected by two separation properties satisfied by the topology $\mathcal{T}_{E,m^*}$.

Firstly, given two events X and Y, if X and Y are two distinct terms of a same sequence $(K^m(E'_\omega))_{m>0}$ for some $\omega \in \Omega$, and both are iterated mutual knowledge of order strictly smaller than m^* in this sequence, then X and Y are T_2-separable, and therefore also T_0-separable.[4] Secondly, if X and Y are two different elements of a same sequence $(K^m(E'_\omega))_{m>0}$ for some $\omega \in \Omega$, and both are iterated mutual knowledge of order strictly larger than m^* in this sequence, then X and Y are T_0-separable, yet not T_2-separable. According to these two separation properties, agents have access to a more refined distinction between the m^* first iterated knowledge claims of their posterior beliefs in E than between the iterated mutual knowledge claims of order strictly larger than m^*. In other words, iterated mutual knowledge claims are only precisely discerned up to a given amount of iterations, and thereafter the higher iterations become less distinguishable for the agents. Also, from a bounded rationality point of view, the agent perception of higher-order mutual knowledge furnished by the topology $\mathcal{T}_{E,m^*}$ reflects that people typically lose track from some iteration level onwards when reasoning about higher-order mutual knowledge.

Farther, the topology $\mathcal{T}_{E,m^*}$ notably satisfies the following epistemic-topological property: for any event E'_ω, if the sequence $(K^m(E'_\omega))_{m>0}$ is strictly shrinking, then $LK(E'_\omega) = K^{m^*}(E'_\omega)$. Indeed, suppose that the sequence $(K^m(E'_\omega))_{m>0}$ is strictly shrinking. Then, by definition of $\mathcal{T}_{E,m^*}$, the only open neighbourhoods of $K^{m^*}(E'_\omega)$ are $\mathcal{P}(\Omega)$ and $\{K^m(E'_\omega) : m \geq 0\}$. Since both sets contain all terms of the sequence $(K^m(E'_\omega))_{m>0}$, it follows that $K^{m^*}(E'_\omega)$ is a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. To see that this limit point is actually unique, consider $F \in \mathcal{P}(\Omega)$ such that $F \neq K^{m^*}(E'_\omega)$. Then either $F = K^m(E'_{\omega'})$ for some $m < m^*$ and some $\omega' \in \Omega$, or $F = K^m(E'_{\omega'})$ for some $m > m^*$ and some $\omega' \in \Omega$, or $F = K^{m^*}(E'_{\omega'})$ for some $\omega' \neq \omega$, or $F \neq K^m(E'_{\omega'})$ for all $m \geq 0$ and all $\omega' \in \Omega$. These four mutually exclusive cases are now considered in turn. First of all, if $F = K^m(E'_{\omega'})$ for some $m < m^*$ and some $\omega' \in \Omega$, then $\{K^m(E'_{\omega'})\}$ is an open neighbourhood of F. Since the sequence $(K^m(E'_\omega))_{m>0}$ is strictly shrinking, it can then not be the case that the singleton open neighbourhood $\{K^m(E'_{\omega'})\}$ of F contains all terms of the sequence $(K^m(E'_\omega))_{m>0}$ from some index onwards. Therefore F is not a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. Secondly, if $F = K^m(E'_{\omega'})$ for some $m > m^*$ and some $\omega' \in \Omega$, then $\{K^{m^*+j}(E'_{\omega'}) : 0 < j \leq m - m^*\}$ is an open neighbourhood of F. Since the set $\{K^{m^*+j}(E'_{\omega'}) : 0 < j \leq m - m^*\}$ is finite, F cannot be a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. Thirdly, if $F = K^{m^*}(E'_{\omega'})$ for some

[4] Given a topological space $(A, \mathcal{T})$, two points in A are called T_2-*separable* if there exist two disjoint $\mathcal{T}$-open neighbourhoods of these two points. Moreover, two points in A are called T_0-*separable* if there exists a $\mathcal{T}$-open set containing precisely one of these two points. Note that T_2-separability implies T_0-separability.

$\omega' \neq \omega$, then $\{K^n(E'_{\omega'}) : n \geq 0\}$ is an open neighbourhood of F. Moreover, since $K^{m^*}(E'_\omega) \neq K^{m^*}(E'_{\omega'}) = F$, it directly follows that $E'_\omega \neq E'_{\omega'}$. Yet since $\mathcal{C} = \{E'_{\omega''} : \omega'' \in \Omega\}$ is a partition of Ω, it holds that $E'_\omega \cap E'_{\omega'} = \emptyset$. Moreover, as $K^m(E'_\omega) \subseteq E'_\omega$ for all $m \geq 0$, and $K^n(E'_{\omega'}) \subseteq E'_{\omega'}$ for all $n \geq 0$, as well as $E'_\omega \cap E'_{\omega'} = \emptyset$, it follows that $K^m(E'_\omega) \neq K^n(E'_{\omega'})$ for all $m, n \geq 0$. Thus the open neighbourhood $\{K^n(E'_{\omega'}) : n \geq 0\}$ of F contains no term of the sequence $(K^m(E'_\omega))_{m>0}$ whatsoever. Therefore, F is not a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. Fourthly, if $F \neq K^m(E'_{\omega'})$ for all $m \geq 0$ and all $\omega' \in \Omega$, then $\mathcal{P}(\Omega) \setminus \{K^m(E'_\omega) : m \geq 0 \text{ and } \omega \in \Omega\}$ is an open neighbourhood of F. Yet this set contains no term of the sequence $(K^m(E'_\omega))_{m>0}$. Thus F is not a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. To summarize, there consequently exists no $F \neq K^{m^*}(E'_\omega)$ which is a limit point of the sequence $(K^m(E'_\omega))_{m>0}$. Therefore, the limit point $K^{m^*}(E'_\omega)$ of the sequence $(K^m(E'_\omega))_{m>0}$ is unique, and thence $LK(E'_\omega) = \lim_{m\to\infty} K^m(E'_\omega) = K^{m^*}(E'_\omega)$. Furthermore, since the sequence $(K^m(E'_\omega))_{m>0}$ is strictly shrinking, $CK(E'_\omega) = \bigcap_{m>0} K^m(E'_\omega) \subsetneq K^{m^*}(E'_\omega)$, and hence $CK(E'_\omega) \neq LK(E'_\omega)$.

Finally, the following example describes an interactive situation, in which the epistemically-based topology $\mathcal{T}_{E,m^*}$ provides a possibility for the agents to agree to disagree on their posterior beliefs with limit knowledge.

Example 1. Consider the Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i\in I}, p)$, where $\Omega = \{\omega_n : n \geq 0\}$, $I = \{Alice, Bob\}$, $I_{Alice} = \{\{\omega_0\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\} \cup \{\{\omega_{2n}, \omega_{2n+1}\} : n \geq 5\}$, $I_{Bob} = \{\{\omega_0, \omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \cup \{\{\omega_{2n+1}, \omega_{2n+2}\} : n \geq 4\}$, and $p : \Omega \to \mathbb{R}$ is given by $p(\omega_n) = \frac{1}{2^{n+1}}$ for all $n \geq 0$. Also, consider the event $E = \{\omega_1, \omega_5\} \cup \{\omega_{2n} : n \geq 1\}$ and the world ω_{10}. Besides, for sake of notational convenience, let the event $\bigcap_{i\in I}\{\omega' \in \Omega : p(E \mid \mathcal{I}_i(\omega')) = p(E \mid \mathcal{I}_i(\omega_{10}))\}$ be denoted by E'. First of all, observe that the computation of the posterior beliefs of *Alice* and *Bob* gives a variety of distinct values for the first ten worlds $\{\omega_0, \omega_1, \ldots, \omega_9\}$, as well as $p(E \mid \mathcal{I}_{Alice}(\omega_n)) = \frac{2}{3}$ and $p(E \mid \mathcal{I}_{Bob}(\omega_n)) = \frac{1}{3}$, for all $n \geq 10$. It follows that $\{\omega' \in \Omega : p(E \mid \mathcal{I}_{Alice}(\omega')) = p(E \mid \mathcal{I}_{Alice}(\omega_{10}))\} = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_9\}$ and $\{\omega' \in \Omega : p(E \mid \mathcal{I}_{Bob}(\omega')) = p(E \mid \mathcal{I}_{Bob}(\omega_{10}))\} = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_8\}$, thus $E' = (\Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_9\}) \cap (\Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_8\}) = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_9\}$. Moreover, the definitions of the possibility partitions of *Alice* and *Bob* ensure that $K^m(E') = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_{m+9}\}$, for all $m > 0$. Consequently, the sequence $(K^m(E'))_{m>0}$ is strictly shrinking and $CK(E') = \bigcap_{m>0} K^m(E') = \emptyset$. Now, let $m^* > 0$ be some index and suppose that $\mathcal{P}(\Omega)$ is equipped with the topology $\mathcal{T}_{E,m^*}$. Since the sequence $(K^m(E'))_{m>0}$ is strictly shrinking, the definition of this topology ensures that $LK(E') = K^{m^*}(E') = \Omega \setminus \{\omega_0, \omega_1, \ldots, \omega_{m^*+9}\}$. Consequently, the computations of the posterior beliefs of *Alice* and *Bob* give $p(E \mid \mathcal{I}_{Alice}(\omega)) = \frac{2}{3}$ and $p(E \mid \mathcal{I}_{Bob}(\omega)) = \frac{1}{3}$, for all $\omega \in LK(E')$. In other words, for all $\omega \in LK(E')$, it holds that $p(E \mid \mathcal{I}_{Alice}(\omega)) \neq p(E \mid \mathcal{I}_{Bob}(\omega))$.

4 Conclusion

In an epistemic-topoloigcal framework, agents have been shown to be able to agree to disagree. More precisely, if Bayesian agents entertain a common prior belief in a given event as well as limit knowledge of their posterior beliefs in the event, then their actual posterior beliefs may indeed differ. This possibility result also holds in interactive situations enriched by a particular epistemically-based topology revealing a cogent agent perception of the event space.

The topological approach to set-based interactive epistemology, in which topologies model agent closeness perceptions of events, can be used to describe various agent reasoning patterns that do not only depend on mere epistemic but also on topological features of the underlying interactive situation. For instance, the event *It is cloudy in London* seems to be closer to the event *It is raining in London* than the event *It is sunny in London*. Now, agents may make identical decisions only being informed of the truth of some event within a class of *close* events. Indeed, *Alice* might decide to stay at home not only in the case of it raining outside, but also in the case of events perceived by her to be similar such as it being cloudy outside.

Moreover, we envision the construction of a more general epistemic-topological framework – *topological Aumann structures* – comprising topologies not only on the event space but also on the state space. Such an extension permits an explicit consideration of a notion of closeness between events as well as between worlds, enabling to model common agent perceptions of the event and state spaces. In particular, it might be of distinguished interest to base topologies on first principles such as epistemic axioms or natural closeness properties. In line with this perspective, the topology provided in Section 3 reflects the natural agent perception for which iterated mutual knowledge becomes imprecise from some level onwards.

Besides, in order to model subjective rather than common agent perceptions of the event and state spaces, the epistemic-topological framework envisioned here could be amended by assigning specific and potentially distinct topologies to every agent. A collective topology reflecting a common closeness perception could then be constructed on the basis of the particular agent topologies, and limit knowledge be defined in such a global topological context. For instance, by providing a topology that is coarser than each agent's one, the meet topology could be used as a representative collective topology. Alternatively, an agent specific operator limit knowledge could be defined with respect to each particular topology, and mutual limit knowledge as their intersection then be considered.

Finally, in a general epistemic-topological framework, various issues can be addressed. For example, the possibility of agents to agree to disagree with limit knowledge can be further analyzed for other epistemically-based as well as agent specific topologies. Furthermore, analogously to the epistemic program in game theory that attempts to provide epistemic foundations for solution concepts, an epistemic-topological approach could generate epistemic-topological foundations for solution concepts. In addition, it could be attempted to develop a theory of counterfactuals in set-based interactive epistemology founded on a notion of similarity of worlds or events furnished by topologies on the state or event space, respectively.

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