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Common knowledge and limit knowledge

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Abstract We study the relationship between common knowledge and the sequence of iterated mutual knowledge from a topological point of view. It is shown that common knowledge is not equivalent to the limit of the sequence of iterated mutual knowledge. On that account the new epistemic operator limit knowledge is introduced and analyzed in the context of games. Indeed, an example is constructed where the behavioral implications of limit knowledge of rationality strictly refine those of common knowledge of rationality. More generally, it is then shown that limit knowledge of rationality is capable of characterizing any solution concept for some appropriate epistemic-topological conditions. Finally, some perspectives of a topologically enriched epistemic framework for games are discussed.

Keywords Aumann structures · Common knowledge · Epistemic game theory · Interactive epistemology · Limit knowledge

1 Introduction

Interactive epistemology provides a general framework in which epistemic notions such as knowledge and belief can be modelled for situations involving multiple agents.

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This rather recent discipline has been initiated by Aumann (1976) and first been adopted in the particular context of games by Aumann (1987) as well as by Tan and Werlang (1988). The objectives of an epistemic approach to game theory consists in characterizing existing solution concepts in terms of epistemic assumptions, as well as in proposing new solution concepts by studying the consequences of refined or novel epistemic hypotheses. Actually, epistemic game theory can be regarded as complementing classical game theory. While the latter is based on the two basic primitives—game form and choice—the former adds an epistemic framework as a third elementary component such that knowledge and beliefs can be explicitly modelled in games. Here, we follow Aumann's set-based approach to epistemic game theory, as introduced in Aumann (1976) and developed notably by Aumann (1987, 1995, 1996, 1998a,b, 1999a,b) and Aumann (2005).

A central epistemic concept in game theory is common knowledge. It is used in basic background assumptions, such as common knowledge of the game form, as well as in epistemic hypotheses, such as common knowledge of rationality, which in turn can be applied to epistemic foundations of solution concepts. Originally, the notion has been introduced by Lewis (1969) as a prerequisite for a rule to become a convention. Intuitively, some event is said to be common knowledge among a set of agents, if everyone knows the event, everyone knows that everyone knows the event, everyone knows that everyone knows that everyone knows the event, etc. Indeed, it has become standard to define common knowledge as the infinite intersection, or conjunction, of iterated mutual knowledge claims. Alternatively, it is possible to conceive of common knowledge as a fixed point by defining common knowledge of some event as the claim that everyone knows both the event and common knowledge of the event.[1] The natural question then arises whether these two definitions are equivalent. In fact, Barwise (1988) provides a special situation-theoretic model in which the standard and fixed point views of common knowledge do not coincide. Moreover, van Benthem and Sarenac (2004) also show the non-equivalence of the two notions within a framework of epistemic logic with topological semantics.

Apart from comparing distinct conceptions of common knowledge, a further intriguing and somewhat related question that can be addressed concerns the relationship between the standard definition of common knowledge and the infinite sequence of iterated mutual knowledge underlying it. Indeed, Lipman (1994) shows that common knowledge of the particular event rationality is not equivalent to the limit of iterated mutual knowledge for some specific notion of limit. Here, we also study the relationship between common knowledge and the limit of the sequence of iterated mutual knowledge, but from a topological point of view. More precisely, it is shown that common knowledge is not equivalent to the limit of the sequence of iterated mutual knowledge, and on that account the new epistemic operator *limit knowledge* is introduced as well as analyzed in the context of games.

We proceed as follows. In Sect. 2, the basic framework of set-based epistemic game theory is presented and the standard definition of common knowledge stated. Besides, an epistemic foundation for iterated strict dominance involving a weaker

[1] Note that such a fixed point view is also implicit in Aumann (1976) meet definition of common knowledge.

than standard notion of rationality is given for possibly infinite games. Furthermore, Sect. 3 studies the relationship between common knowledge and the sequence of iterated mutual knowledge from a topological point of view, shows that the concept of common knowledge genuinely differs from the former's limit, and defines the new epistemic operator limit knowledge. Next, Sect. 4 studies some game-theoretic consequences of limit knowledge of the specific event rationality. In particular, a concrete static infinite game is constructed in which limit knowledge of rationality strictly refines common knowledge of rationality in terms of solution concepts. In this example, the latter epistemic hypothesis implies iterated strict dominance, while the former entails iterated strict dominance followed by weak dominance. It is then generally shown that, for any given game and epistemic model of it satisfying some appropriate epistemic-topological conditions, limit knowledge of rationality is capable of characterizing every solution concept. Due to this universal foundational capability, limit knowledge of rationality could thus be used for epistemic-topological characterizations of solution concepts. Moreover, Sect. 5 discusses some perspectives of a general topological framework for set-based interactive epistemology. Finally, Sect. 6 offers some concluding remarks.

2 Common knowledge

Before common knowledge is defined formally, the set-based framework for interactive epistemology is briefly presented. A so-called Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i\in I})$ contains a set Ω of possible worlds, which are complete descriptions of the way the world might be, and an information partition $\mathcal{I}_i$ of Ω for each agent $i \in I$. The cell of $\mathcal{I}_i$ containing the world ω is denoted by $\mathcal{I}_i(\omega)$ and assembles all worlds considered possible by i at world ω. Intuitively, an agent i cannot distinguish between any two worlds ω and ω' that are in the same cell of his partition $\mathcal{I}_i$. Two such worlds are called indistinguishable for agent i. Alternatively, if $\mathcal{I}_i(\omega) \neq \mathcal{I}_i(\omega')$, then ω and ω' are said to be distinguishable for agent i. We then call two worlds ω and ω' distinguishable if they are distinguishable for all agents $i \in I$. Moreover, an event $E \subseteq \Omega$ is defined as a set of possible worlds. For example, the event of it raining in London contains all worlds in which it does in fact rain in London. Farther, an Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i\in I})$ is called finite if Ω is finite and infinite otherwise.

In Aumann structures knowledge is formalized in terms of events. Indeed, the event that agent i knows some event E, denoted by $K_i(E)$, is defined as $K_i(E) := \{\omega \in \Omega : \mathcal{I}_i(\omega) \subseteq E\}$. Intuitively, i knows some event E if in all worlds he considers possible E holds. If $\omega \in K_i(E)$, then i is said to know E at world ω. Naturally, the event $K(E) = \bigcap_{i\in I} K_i(E)$ denotes mutual knowledge of E among the set I of agents. Iterated mutual knowledge can then be formalized inductively. More precisely, letting $K^0(E) := E$, m-order mutual knowledge of the event E among the set I of agents is defined by $K^m(E) := K(K^{m-1}(E))$ for all $m > 0$. Accordingly, mutual knowledge can also be denoted as 1-order mutual knowledge. Different iterated mutual knowledge claims are related by the following lemma.

Lemma 1 *Let $\mathcal{A}$ be an Aumann structure and E be some event. For all $m' \geq m \geq 0$, $K^{m'}(E) \subseteq K^m(E)$.*

Proof The proof is by induction on m'. First of all, if $m' = 0$, then $m = m' = 0$, and thus $K^{m'}(E) \subseteq K^m(E)$. Now, assume that $m' = p' + 1$ for some $p' \geq 0$, and that $K^{p'}(E) \subseteq K^p(E)$ for all p such that $p' \geq p \geq 0$. If $m = m'$, then $K^{m'}(E) \subseteq K^m(E)$. If $m < m'$, then $m \leq p'$, and hence by the induction hypothesis, and since the mutual knowledge operator K is monotone with respect to set inclusion, it follows that $K^{m'}(E) = K^{p'+1}(E) = K(K^{p'}(E)) \subseteq K(K^m(E)) \subseteq K^m(E)$. ■

In fact, Lemma 1 generalizes the characteristic property of knowledge, the so-called truth axiom $K_i(E) \subseteq E$, to arbitrary higher-order mutual knowledge. The notable contrast between knowledge and belief resides in the very fact that false claims cannot be known, yet can be believed. Moreover, by Lemma 1, any sequence of iterated mutual knowledge $(K^m(E))_{m>0}$ can be concluded to be either *strictly shrinking*, i.e., $K^{m+1}(E) \subsetneq K^m(E)$ for all $m \geq 0$, or *eventually constant*, i.e., there exists some index p such that $K^m(E) = K^p(E)$ for all $m \geq p$. The case of sequences of iterated mutual knowledge being strictly shrinking will be of specific importance in the sequel.

Besides, an event is said to be common knowledge among a set I of agents whenever all m-order mutual knowledge simultaneously hold. The standard definition formalizes the concept as follows.

Definition 1 Let $\mathcal{A}$ be an Aumann structure and E be some event. $\mathrm{CK}(E) := \bigcap_{m>0} K^m(E)$ is the event that E is *common knowledge*.

Common knowledge of the particular event that all players are rational has been used in epistemic characterizations of solution concepts in games. A well-known result, e.g., Bernheim (1984), Pearce (1984), Tan and Werlang (1988), as well as Börgers (1993), states that iterated strict dominance is epistemically characterized by common knowledge of rationality with the standard notion of rationality as subjective expected utility maximization. We now give an epistemic foundation of pure strategy iterated strict dominance in terms of common knowledge of some weaker than standard rationality for possibly infinite games. More precisely, we employ a normal form adapted version of Aumann (1995) knowledge-based notion of rationality, originally stated for extensive forms with perfect information. As argued by Aumann (1995) and Aumann (1996), this notion more general and simpler than standard subjective expected utility maximization, since the latter implies the former but the former does not imply the latter, and knowledge-based rationality completely dispenses with probabilities.

First, some standard game-theoretic notation and notions are recalled. In the sequel, $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ denotes an arbitrary game, i.e., with possibly infinitely many players and possibly infinite strategy spaces, and $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i \in I}, (\sigma_i)_{i \in I})$ an epistemic model of it. When being employed in the context of games, an Aumann structure additionally specifies a choice function $\sigma_i : \Omega \to S_i$ for each player $i \in I$ that connects the interactive epistemology to the game. The choice function profile $\sigma : \Omega \to \times_{i \in I} S_i$ mapping each world to its corresponding strategy profile is then defined by $\sigma(\omega) = (\sigma_i(\omega))_{i \in I}$. Moreover, it is standard to assume that each player knows his own strategy choice. This so-called measurability assumption seems natural in the context of game theory, where agents make their choices deliberately and consicously. Aumann and Brandenburger (1995) even denote it as tautologous by pointing out that knowing one's own choice is implicit in consciously making a choice. Formally, the measurability assumption requires each player's choice function σ_i to be measurable with respect to $\mathcal{I}_i$, i.e., if two worlds ω and ω' are in the same cell of player i's information partition, then $\sigma_i(\omega) = \sigma_i(\omega')$.

Next, the weaker than standard notion of rationality used in the sequel is defined.

Definition 2 Let Γ be a game, $\mathcal{A}^\Gamma$ be an epistemic model of it, and i be some player. The event that i is *rational* is defined as

$$R_i := \bigcap_{s_i \in S_i} (\Omega \setminus K_i(\{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega)) > u_i(\sigma(\omega))\}))\,.$$

Accordingly, a player i is rational—in a weak sense—whenever for any of his strategies $s_i \in S_i$, he does not know that s_i would yield him higher utility than his actual choice. In other words, i is rational at ω if for any of his strategies $s_i \in S_i$ he considers possible a world $\omega' \in \mathcal{I}_i(\omega)$ in which his strategy choice $\sigma_i(\omega')$, being equal to his actual choice $\sigma_i(\omega)$ by measurability, could give him at least as much utility as s_i. The event $R := \bigcap_{i \in I} R_i$ that all players are rational is called *rationality*.

In game theory, so-called solution concepts are developed that reduce the strategy profile space. Formally, a solution concept $\mathcal{SC}$ consists of a mapping associating with each game Γ a subset $\mathcal{SC}^\Gamma \subseteq \times_{i \in I} S_i$ of its strategy profile space. A solution concept thus provides a generic method which does not depend on any particular given game. Intuitively, a solution concept yields the choices a player should make. One of the most established game-theoretic solution concepts for the normal form is iterated strict dominance, which can be defined as follows.

Definition 3 Let $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ be a game. Moreover, let $S_i^0 = S_i$ for all $i \in I$, and let the sequence $(\mathrm{SD}^k)_{k \geq 0}$ of strategy profile sets be inductively given by $\mathrm{SD}^0 = \times_{i \in I} S_i^0$ and $\mathrm{SD}^{k+1} = \times_{i \in I} \mathrm{SD}_i^{k+1}$, where $\mathrm{SD}_i^{k+1} = \mathrm{SD}_i^k \setminus \{s_i \in \mathrm{SD}_i^k :$ there exists $s_i' \in \mathrm{SD}_i^k$ such that $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$, for all $s_{-i} \in \mathrm{SD}_{-i}^k\}$, for all $i \in I$. The solution concept *iterated strict dominance* is defined as $\mathrm{ISD}^\Gamma := \bigcap_{k \geq 0} \mathrm{SD}^k$.

Note that Dufwenberg and Stegeman (2002) study iterated strict dominance for arbitrary static games in a non-epistemic context, unveiling potential ill-behaviour. It is shown that iterated strict dominance can be order-dependent, return an empty set of strategy profiles, or fail to yield a maximal reduction after countably many steps. Moreover, they prove the existence and uniqueness of a non-empty maximal reduction by requiring compactness of the players' strategy spaces and continuity of the utility functions. However, according to Definition 3, order dependence is no longer a possible problem, since at each round, all the remaining strictly dominated strategies are eliminated.

We now give an epistemic foundation of pure strategy iterated strict dominance in terms of common knowledge of rationality for possibly infinite games with the weaker than standard concept of knowledge-based rationality. Note that in Proposition 1 below, as well as in all results of Sect. 4, common knowledge of the structure of the game is endorsed as an implicit background assumption.

Proposition 1 *Let Γ be a game and $\mathcal{A}^\Gamma$ be an epistemic model of it. Then, $\sigma(CK(R)) \subseteq ISD^\Gamma$.*

Proof By induction, we prove that $\sigma(K^m(R)) \subseteq \mathrm{SD}^{m+1}$, for all $m \geq 0$. It then follows that $\sigma(\mathrm{CK}(R)) = \sigma(\bigcap_{m>0} K^m(R)) = \sigma(\bigcap_{m\geq 0} K^m(R)) \subseteq \bigcap_{m\geq 0} \sigma(K^m(R)) \subseteq \bigcap_{m\geq 0} \mathrm{SD}^{m+1} = \bigcap_{m>0} \mathrm{SD}^m = \bigcap_{m\geq 0} \mathrm{SD}^m = \mathrm{ISD}^\Gamma$. First of all, consider $(s_i)_{i\in I} \in \sigma(K^0(R)) = \sigma(R)$. Then, there exists $\omega \in R = \bigcap_{i\in I} R_i$ such that $\sigma(\omega) = (s_i)_{i\in I}$. Hence, by definition of R_i and measurability of σ_i, for all $s_i \in S_i$, there exists $\omega' \in \mathcal{I}_i(\omega)$ such that $u_i(s_i, \sigma_{-i}(\omega')) \leq u_i(\sigma(\omega')) = u_i(\sigma_i(\omega), \sigma_{-i}(\omega'))$. It follows that $\sigma_i(\omega) \in \mathrm{SD}_i^1$ for all $i \in I$, and thus $\sigma(\omega) = (s_i)_{i\in I} \in \times_{i\in I}\mathrm{SD}_i^1 = \mathrm{SD}^1$. Therefore, $\sigma(K^0(R)) \subseteq \mathrm{SD}^1$ obtains. Now, assume $\sigma(K^m(R)) \subseteq \mathrm{SD}^{m+1}$ for some $m > 0$, and let $(s_i)_{i\in I} \in \sigma(K^{m+1}(R))$. Then, there exists $\omega \in K^{m+1}(R)$ such that $\sigma(\omega) = (s_i)_{i\in I}$. Hence, $\mathcal{I}_i(\omega) \subseteq K^m(R)$, and thus, by the induction hypothesis, $\sigma(\mathcal{I}_i(\omega)) \subseteq \mathrm{SD}^{m+1}$ obtains. Besides, since $\omega \in R_i$, for all $s_i \in \mathrm{SD}_i^{m+1}$ there exists $\omega' \in \mathcal{I}_i(\omega)$ such that $u_i(s_i, \sigma_{-i}(\omega')) \leq u_i(\sigma(\omega')) = u_i(\sigma_i(\omega), \sigma_{-i}(\omega'))$. Yet since $\sigma(\mathcal{I}_i(\omega)) \subseteq \mathrm{SD}^{m+1}$, each $\omega' \in \mathcal{I}_i(\omega)$ induces $\sigma_{-i}(\omega') \in \mathrm{SD}_{-i}^{m+1}$, which in turn implies that $\sigma_i(\omega) \in \mathrm{SD}_i^{m+2}$ for all $i \in I$, and consequently $(s_i)_{i\in I} = \sigma(\omega) \in \times_{i\in I}\mathrm{SD}_i^{m+2} = \mathrm{SD}^{m+2}$. Therefore, $\sigma(K^{m+1}(R)) \subseteq \mathrm{SD}^{m+2}$ holds, which concludes the proof. ■

3 Limit knowledge

The sequence of iterated mutual knowledge constitutes the essential ingredient of common knowledge. Indeed, according to the standard definition, common knowledge of an event is the countably infinite intersection of all successive higher-order mutual knowledge of the event. Thence, a natural question to be addressed is to clarify the relationship between common knowledge and the possible limit points of the sequence of higher-order mutual knowledge, from a topological point of view. In fact, we show that these two concepts are closely related for finite, yet do substantially differ for infinite Aumann structures.

First of all, for any finite Aumann structure and any topology on the event space, common knowledge of an event E is always a limit point of the sequence of higher-order mutual knowledge of E, as established by the following result.

Proposition 2 *Let $\mathcal{A}$ be a finite Aumann structure, $\mathcal{T}$ be a topology on $\mathcal{P}(\Omega)$, and E be some event. Then, CK(E) is a limit point of $(K^m(E))_{m>0}$.*

Proof Note that since Ω is finite, its power set $\mathcal{P}(\Omega)$ is also finite. Moreover, Lemma 1 ensures that $K^{m+1}(E) \subseteq K^m(E)$ for all $m > 0$. Thus, by finiteness of $\mathcal{P}(\Omega)$, the sequence $(K^m(E))_{m>0}$ is eventually constant, i.e., there exists some index p such that $K^m(E) = K^p(E)$ for all $m \geq p$. Thence $\mathrm{CK}(E) = \bigcap_{m>0} K^m(E) = \bigcap_{m\geq p} K^m(E) = K^p(E)$. Moreover, for any $\mathcal{T}$-open neighbourhood N of $\mathrm{CK}(E)$, it holds that $K^m(E) = K^p(E) = \mathrm{CK}(E) \in N$, for all $m \geq p$. Therefore, $\mathrm{CK}(E)$ is a limit point of the sequence $(K^m(E))_{m>0}$. ■

Note that the sequence $(K^m(E))_{m>0}$ may converge to multiple limit points, $\mathrm{CK}(E)$ always being one of them. In particular, if $\mathcal{P}(\Omega)$ is equipped with a Hausdorff topology, then $\mathrm{CK}(E)$ is equal to the unique limit of $(K^m(E))_{m>0}$. Since the discrete topology is the only Hausdorff topology available for finite spaces, the event space $\mathcal{P}(\Omega)$ being equipped with the discrete topology ensures that $\lim_{m\to\infty} K^m(E) = \mathrm{CK}(E)$.

Now, infinite Aumann structures are considered. In this case, the following result shows that, if the sequence of iterated mutual knowledge is eventually constant, then common knowledge is always one of its limit points.

Proposition 3 *Let $\mathcal{A}$ be an infinite Aumann structure, $\mathcal{T}$ be a topology on $\mathcal{P}(\Omega)$, and E be some event. If $(K^m(E))_{m>0}$ is eventually constant, then $CK(E)$ is a limit point of $(K^m(E))_{m>0}$.*

Proof Suppose that the sequence $(K^m(E))_{m>0}$ is eventually constant from index p onwards. By Lemma 1, it follows that $\mathrm{CK}(E) = \bigcap_{m>0} K^m(E) = \bigcap_{m>p} K^m(E) = K^p(E)$. Now let N be a $\mathcal{T}$-open neighborhood of $\mathrm{CK}(E)$. Since both $K^m(E) = K^p(E)$ for all $m \geq p$ and $K^p(E) = \mathrm{CK}(E)$, it follows that $K^m(E) \in N$ for all $m \geq p$. Therefore, $\mathrm{CK}(E)$ is a limit point of the sequence $(K^m(E))_{m>0}$. ■

Accordingly, it follows that common knowledge and the limit of iterated mutual knowledge can only possibly be distinct in the case of the sequence of iterated mutual knowledge not being eventually constant. Since the latter sequence either is eventually constant or strictly shrinking, potential differences of the two concepts necessarily require the strictly shrinking condition to be met.

In case of the sequence of iterated mutual knowledge being strictly shrinking, common knowledge and its topological limit may indeed differ.

Proposition 4 *There exist an infinite Aumann structure $\mathcal{A}$, a topology on the event space $\mathcal{P}(\Omega)$, and some event E, such that $CK(E) \neq \lim_{m\to\infty} K^m(E)$.*

Proof Consider the infinite Aumann structure $\mathcal{A} = (\mathbb{N}, (\mathcal{I}_i)_{i\in\{\text{Alice},\text{Bob}\}})$ given by

$$\begin{aligned}\mathcal{I}_{\text{Alice}} &= \{\{0\}, \{1,3\}, \{2,4\}, \{5,7\}, \{6,8\}, \{9,11\}, \{10,12\}, \ldots\},\\ \mathcal{I}_{\text{Bob}} &= \{\{0,2\}, \{1\}, \{3,5\}, \{4,6\}, \{7,9\}, \{8,10\}, \{11,13\}, \ldots\}.\end{aligned}$$

Let E be the event $\mathbb{N} \setminus \{0\}$. Then, $K_{\text{Alice}}(E) = \mathbb{N} \setminus \{0\}$ and $K_{\text{Bob}}(E) = \mathbb{N} \setminus \{0, 2\}$, thus $K^1(E) = K(E) = K_{\text{Alice}}(E) \cap K_{\text{Bob}}(E) = \mathbb{N} \setminus \{0, 2\}$. It follows by induction that $K^m(E) = \mathbb{N} \setminus \{0, 2, \ldots, 2m\}$ for all $m > 0$. Consequently $K^{m+1}(E) \subsetneq K^m(E)$ for all $m > 0$, i.e., the sequence $(K^m(E))_{m>0}$ is strictly shrinking. Moreover, $\mathrm{CK}(E) = \bigcap_{m>0} K^m(E) = \bigcap_{m>0} \mathbb{N} \setminus \{0, 2, \ldots, 2m\} = \{2n + 1 : n \geq 0\}$. Now, let $L \subseteq \Omega$ be some event different from $\mathrm{CK}(E)$, and suppose that the event space $\mathcal{P}(\mathbb{N})$ is equipped with the topology $\mathcal{T} = \{O \subseteq \mathcal{P}(\mathbb{N}) : L \notin O\} \cup \{\mathcal{P}(\mathbb{N})\}$. Then, the only $\mathcal{T}$-open neighbourhood of L is $\mathcal{P}(\mathbb{N})$, and, since all terms of the sequence $(K^m(E))_{m>0}$ are contained in $\mathcal{P}(\mathbb{N})$, it follows that L is a limit point of the sequence $(K^m(E))_{m>0}$. Moreover, L is actually the unique limit point of $(K^m(E))_{m>0}$. Indeed, since $(K^m(E))_{m>0}$ satisfies the strictly shrinking condition, for any event $F \neq L$, the elements of $(K^m(E))_{m>0}$ will never all be contained in the $\mathcal{T}$-open neighbourhood $\{F\}$ of F from some index onwards, showing that F is not a limit point of $(K^m(E))_{m>0}$. Hence, $\lim_{m\to\infty} K^m(E) = L$. Yet since L was precisely chosen to be different from $\mathrm{CK}(E)$, it follows that $\lim_{m\to\infty} K^m(E) \neq \mathrm{CK}(E)$. ■

The following example shows that common knowledge may even differ from the unique limit of the sequence of higher-order mutual knowledge in the case of so-called well-behaved—i.e., completely metrizable and Hausdorff—topologies.

Example 1 Let $\mathcal{A} = (\mathbb{N}, \mathcal{I}_{\text{Alice}}, \mathcal{I}_{\text{Bob}})$ be the infinite Aumann structure described in the proof of Proposition 4, and E be the event $\mathbb{N} \setminus \{0\}$. Then, as shown in the proof of Proposition 4, $K^m(E) = \mathbb{N} \setminus \{0, 2, \ldots, 2m\}$ and $\text{CK}(E) = \{2n + 1 : n \geq 0\}$. Consider farther the Cantor space $\{0, 1\}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to $\{0, 1\}$ equipped with its usual topology, i.e., the product topology of the discrete topology on $\{0, 1\}$. This space is Polish, i.e., Hausdorff and completely metrizable, and the induced metric is defined by $d(f, g) = 2^{-r}$, where $r = \min\{n : f(n) \neq g(n)\}$. Consider finally the sets $F_1 = \{2n : n \geq 0\}$ and $F_2 = \{2n+1 : n \geq 0\}$, and the bijection $f : \mathcal{P}(\mathbb{N}) \longrightarrow \{0, 1\}^{\mathbb{N}}$ defined by

$$f(F) = \begin{cases} \chi_F & \text{if } F \neq F_1, F_2 \\ \chi_{F_2} & \text{if } F = F_1 \\ \chi_{F_1} & \text{if } F = F_2\,, \end{cases}$$

where χ_A denotes the characteristic function of A. Now, suppose that the event space $\mathcal{P}(\mathbb{N})$ is equipped with the topology $\mathcal{T}$ defined by letting $O \in \mathcal{T}$ if and only if $f(O)$ is an open set of the Cantor space. Since f is an homeomorphism from $\mathcal{P}(\mathbb{N})$ to $\{0, 1\}^{\mathbb{N}}$, the topological space $(\mathcal{P}(\mathbb{N}), \mathcal{T})$ is also Polish, and hence every sequence converges to at most one limit point. We next prove that the sequence $(\chi_{K^m(E)})_{m>0}$ converges to χ_{F_2} in the Cantor space $\{0, 1\}^{\mathbb{N}}$. First of all, the proof of Proposition 4 ensures that $\chi_{K^m(E)} = \chi_{\mathbb{N}\setminus\{0,2,\ldots,2m\}}$ for all $m > 0$. Moreover, observe that $d(\chi_{K^m(E)}, \chi_{F_2}) = 2^{-(m+1)}$. Hence, for any $\varepsilon > 0$, it holds that $d(\chi_{K^m(E)}, \chi_{F_2}) = 2^{-(m+1)} < \varepsilon$ for all $m > \log_2(\frac{1}{\varepsilon}) - 1$. Therefore, $\lim_{m\to\infty} \chi_{K^m(E)} = \chi_{F_2}$. Since f is a homeomorphism, it follows that $\lim_{m\to\infty} K^m(E) = \lim_{m\to\infty} f^{-1}(\chi_{K^m(E)}) = f^{-1}(\lim_{m\to\infty} \chi_{K^m(E)}) = f^{-1}(\chi_{F_2}) = F_1 \neq \text{CK}(E)$, yielding the desired property. ■

The existence of situations in which the topological limit of the sequence of iterated mutual knowledge differs from common knowledge motivates the introduction of a novel epistemic concept based on the notion of topological limit. Indeed, *limit knowledge* is defined as follows.

Definition 4 Let $\mathcal{A}$ be an Aumann structure, $\mathcal{T}$ be a topology on the event space $\mathcal{P}(\Omega)$, and E be some event. If the limit point of $(K^m(E))_{m>0}$ is unique, then $\text{LK}(E) := \lim_{m\to\infty} K^m(E)$ is the event that E is *limit knowledge*.

Accordingly, limit knowledge of an event E is constituted by—whenever unique—the limit point of the sequence of iterated mutual knowledge, and thus linked to both epistemic features as well as topological aspects of the event space.

Limit knowledge can be understood as the event which is approached by the sequence of iterated mutual knowledge, according to some notion of closeness between events furnished by a topology on the event space. Thus, the higher the iterated mutual knowledge, the closer this latter epistemic event is to limit knowledge.

Although being more and more proximal to iterated mutual knowledge the higher the iteration, it is possible—depending on the topology —that limit knowledge is not included in all higher-order mutual knowledge or even in the underlying event itself. Therefore, limit knowledge does not a priori inherit the purely epistemic properties of higher-order mutual knowledge or even knowledge. Actually, agents entertaining limit knowledge of some event might notably be in situations in which the event does not hold, while at the same time being arbitrarily close to the highest iterated mutual knowledge of the event.

However, of specific relevance are the situations in which limit knowledge indeed strictly refines common knowledge. In those cases, limit knowledge does imply all iterated mutual knowledge and can be interpreted as some kind of highest iterated mutual knowledge. Note that Example 2 below provides an illustration where limit knowledge is a strict refinement of common knowledge and induces behavioral consequences that cogently differ from the latter.

Besides, even if limit knowledge should not be amalgamated with common knowledge, both operators can be perceived as sharing similar implicative properties with regards to highest iterated mutual knowledge claims. Indeed, while common knowledge bears a standard implicative relation in terms of set inclusion to highest iterated mutual knowledge, limit knowledge can be considered to entertain an implicative relation in terms of set proximity with highest iterated mutual knowledge. Farther, note that limit knowledge becomes interesting as a possible refinement of common knowledge precisely in circumstances of the sequence of iterated mutual knowledge being strictly shrinking, i.e., whenever common knowledge actually requires infinitely many interactive knowledge claims to be computed.

In fact, it is possible to link limit knowledge to topological reasoning patterns of agents based on closeness of events. Indeed, agents satisfying limit knowledge of some event can intuitively be seen to be in a kind of limit situation arbitrarily close to entertaining all highest iterated mutual knowledge of this event, and this situation may influence the agents' reasoning. For instance, since limit knowledge can be regarded as entertaining an implicative relation of proximity with highest iterated mutual knowledge claims, agents being in a situation of limit knowledge and basing their reasoning on closeness of events might therefore infer all highest iterated mutual knowledge claims. In general, note that a reasoning pattern associated with limit knowledge depends on the particular topology on the event space, which fixes the closeness relation between events and thus also determines the limit knowledge event.

Finally, generalizations of the concept of limit knowledge could be conceived to overcome the undefinability of this operator in cases of non unique limit points. For instance, *multiple-limit knowledge of E* could be defined as the union of all limit points of $(K^m(E))_{m>0}$.

4 Limit knowledge of rationality

The new epistemic operator limit knowledge can be used in the context of games. Indeed, we now illustrate that limit knowledge is capable of refining common knowledge in terms of solution concepts. More precisely, a Cournot-type game is constructed where the application of iterated strict dominance followed by weak dominance, denoted by $(\mathrm{ISD}+\mathrm{WD})^\Gamma$ for a given game Γ, is a strict refinement of iterated strict

Donald

		L	R
Claire	U	$(2, 1)$	$(1, 1)$
	D	$(2, 2)$	$(2, 3)$

for all $(x, y) \neq (\frac{1}{3}, \frac{1}{3})$

Donald

		L	R
Claire	U	$(2, 3)$	$(2, 2)$
	D	$(1, 1)$	$(2, 1)$

for $(x, y) = (\frac{1}{3}, \frac{1}{3})$

Fig. 1 The utility function u_{Claire} and u_{Donald} of game γ

dominance.[2] Then, an epistemic Aumann model of this game is given such that the event common knowledge of rationality precisely reveals all the possible strategy profiles that survive iterated strict dominance, while limit knowledge of rationality conveys the unique strategy profile in accordance with iterated strict dominance followed by weak dominance. Moreover, in this case, limit knowledge of rationality being strictly included in common knowledge of rationality is thus being interpretable as some kind of highest iterated mutual knowledge.

Example 2 Consider the game $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ with player set $I = \{\text{Alice}, \text{Bob}, \text{Claire}, \text{Donald}\}$, strategy sets $S_{\text{Alice}} = S_{\text{Bob}} = [0, 1]$, $S_{\text{Claire}} = \{U, D\}$, $S_{\text{Donald}} = \{L, R\}$, and utility functions $u_i : S_{\text{Alice}} \times S_{\text{Bob}} \times S_{\text{Claire}} \times S_{\text{Donald}} \to \mathbb{R}$ for all $i \in I$, defined as $u_{Alice}(x, y, v, w) = x(1 - x - y)$ and $u_{Bob}(x, y, v, w) = y(1 - x - y)$, as well as $u_{Claire}(x, y, v, w)$ and $u_{Donald}(x, y, v, w)$ as given in Fig. 1.

Solving the game by iterated strict dominance—requiring infinitely many elimination rounds—yields the sequence of successively surviving strategy profile sets $\left([a_n, b_n]^2 \times \{U, D\} \times \{L, R\}\right)_{n \geq 0}$, where $[a_0, b_0] = [0, 1]$, $[a_{n+1}, b_{n+1}] = [\frac{a_n + b_n}{2}, b_n]$ if n is odd, and $[a_{n+1}, b_{n+1}] = [a_n, \frac{a_n + b_n}{2}]$ if n is even. The non-unique solution of the game is thus given by the remaining four strategy profiles $\text{ISD}^\Gamma = \bigcap_{n \geq 0} \left([a_n, b_n]^2 \times \{U, D\} \times \{L, R\}\right) = \left\{\frac{1}{3}\right\} \times \left\{\frac{1}{3}\right\} \times \{U, D\} \times \{L, R\}$. However, it is possible to further restrict the remaining strategy sets of *Claire* and *Donald* by a weak dominance argument—a potential refinement that only emerges *after* applying iterated strict dominance. Indeed, in the set ISD^Γ the strategies D and R are weakly dominated by U and L for *Claire* and *Donald*, respectively. Therefore, iterated strict dominance followed by weak dominance yields the singleton set $(\text{ISD} + \text{WD})^\Gamma = \{(\frac{1}{3}, \frac{1}{3}, U, L)\}$ as a possible strictly refined solution of the game.

Before turning towards the epistemic Aumann model of this game, some preliminary observations are needed. Note that, by definition of the utility functions, the best response strategy of *Alice* to an opponents' strategy combination only depends on *Bob*'s choice, and vice versa. More precisely, *Alice*'s and *Bob*'s best response functions $b_{Alice} : [0, 1] \times \{U, D\} \times \{L, R\} \to [0, 1]$ and $b_{Bob} : [0, 1] \times \{U, D\} \times \{L, R\} \to [0, 1]$ are given by $b_{Alice}(y, v, w) = \frac{1-y}{2}$ and $b_{Bob}(x, v, w) = \frac{1-x}{2}$, respectively. Hence, the

[2] Formally, given a game Γ, iterated strict dominance followed by weak dominance is defined as $(\text{ISD} + \text{WD})^\Gamma := \times_{i \in I} \left(\text{ISD}_i^\Gamma \setminus \{s_i \in \text{ISD}_i^\Gamma : \text{there exists } s_i' \in \text{ISD}_i^\Gamma \text{ such that } u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})\right.$ for all $s_{-i} \in \text{ISD}_{-i}^\Gamma$ and $u_i(s_i, s_{-i}') < u_i(s_i', s_{-i}')$ for some $s_{-i}' \in \text{ISD}_{-i}^\Gamma\}\left.\right)$.

set of all strategy profiles in which *Alice* and *Bob* simultaneously play best responses is given by $\{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\}$. On the basis of these two functions, we now describe an infinite sequence $(s^n_{Alice}, s^n_{Bob})_{n \geq 0}$ of strategy combinations for *Alice* and *Bob* which will be central to the construction of our epistemic Aumann model. This sequence is defined for all $n \geq 0$ by induction as follows.

$$\left(s^0_{Alice}, s^0_{Bob}\right) = (0, 1)$$
$$\left(s^1_{Alice}, s^1_{Bob}\right) = \left(0, \frac{1}{2}\right)$$
$$\left(s^{2n+2}_{Alice}, s^{2n+2}_{Bob}\right) = \left(\frac{1 - s^{2n+1}_{Bob}}{2}, s^{2n+1}_{Bob}\right)$$
$$\left(s^{2n+3}_{Alice}, s^{2n+3}_{Bob}\right) = \left(s^{2n+2}_{Alice}, \frac{1 - s^{2n+2}_{Alice}}{2}\right)$$

This infinite sequence $(s^n_{Alice}, s^n_{Bob})_{n \geq 0}$ of strategy combinations for *Alice* and *Bob* is illustrated in Fig. 2. The depicted points indicate its first few elements. Note that the terms $\left(s^{2n+2}_{Alice}, s^{2n+2}_{Bob}\right)$ and $\left(s^{2n+3}_{Alice}, s^{2n+3}_{Bob}\right)$ are given by the projections of their predecessors on *Alice*'s and *Bob*'s best response curves, respectively. Farther, observe that the sequence converges to $\left(\frac{1}{3}, \frac{1}{3}\right)$.

Next, consider the epistemic Aumann model $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i \in I}, (\sigma_i)_{i \in I})$ of Γ, where the countable set of all possible worlds is given by

$$\Omega = \{\alpha, \beta, \gamma, \delta, \alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \ldots\},$$

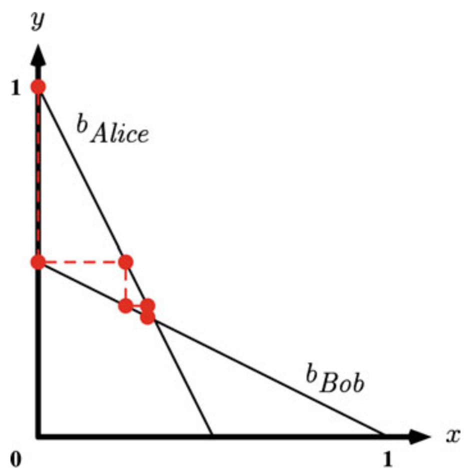

Fig. 2 The infinite sequence $(S^n_{Alice}, S^n_{Bob})_{n \geq 0}$ of strategy combinations for *Alice* and *Bob* is convergent

the players' information partitions are specified by

$$\begin{aligned}
&\mathcal{I}_{Alice} = \{\{\alpha, \beta, \gamma, \delta\}\} \cup \\
&\{\{\alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}, \alpha_{2n+1}, \beta_{2n+1}, \gamma_{2n+1}, \delta_{2n+1}\} : n \geq 0\}, \\
&\mathcal{I}_{Bob} = \{\{\alpha, \beta, \gamma, \delta\}, \{\alpha_0, \beta_0, \gamma_0, \delta_0\}\} \cup \\
&\{\{\alpha_{2n-1}, \beta_{2n-1}, \gamma_{2n-1}, \delta_{2n-1}, \alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}\} : n > 0\}, \\
&\mathcal{I}_{Claire} = \{\{\alpha, \beta\}, \{\gamma, \delta\}\} \cup \\
&\{\{\alpha_n, \beta_n\} : n \geq 0\} \cup \{\{\gamma_n, \delta_n\} : n \geq 0\}, \\
&\mathcal{I}_{Donald} = \{\{\alpha, \gamma\}, \{\beta, \delta\}\} \cup \\
&\{\{\alpha_n, \gamma_n\} : n \geq 0\} \cup \{\{\beta_n, \delta_n\} : n \geq 0\},
\end{aligned}$$

and the choice function $\sigma = (\sigma_{Alice}, \sigma_{Bob}, \sigma_{Claire}, \sigma_{Donald}) : \Omega \to \times_{i \in I} S_i$ assembling all the players' choice functions is defined for all $n \geq 0$ by:

$$\begin{aligned}
\sigma(\alpha) &= (1/3, 1/3, U, L) \quad & \sigma(\alpha_n) &= (s^n_{Alice}, s^n_{Bob}, U, L) \\
\sigma(\beta) &= (1/3, 1/3, U, R) \quad & \sigma(\beta_n) &= (s^n_{Alice}, s^n_{Bob}, U, R) \\
\sigma(\gamma) &= (1/3, 1/3, D, L) \quad & \sigma(\gamma_n) &= (s^n_{Alice}, s^n_{Bob}, D, L) \\
\sigma(\delta) &= (1/3, 1/3, D, R) \quad & \sigma(\delta_n) &= (s^n_{Alice}, s^n_{Bob}, D, R).
\end{aligned}$$

By definition of the sequence $(s^n_{Alice}, s^n_{Bob})_{n \geq 0}$, the two equalities $s^{2n}_{Alice} = s^{2n+1}_{Alice}$ and $s^{2n+1}_{Bob} = s^{2n+2}_{Bob}$ hold for all $n \geq 0$, and therefore our epistemic Aumann model satisfies the standard measurability requirement for the players' choice functions.

We now analyze the players' rationality. First of all, consider *Alice* and note that she is rational at worlds α, β, γ and δ. By construction of the sequence $(s^n_{Alice}, s^n_{Bob})_{n \geq 0}$, if ω is a world such that $(\sigma_{Alice}(\omega), \sigma_{Bob}(\omega)) = (s^{2n}_{Alice}, s^{2n}_{Bob})$ for some $n \geq 0$, then $u_{Alice}(\sigma(\omega)) = u_{Alice}(b_{Alice}(\sigma_{-Alice}(\omega)), \sigma_{-Alice}(\omega)) \geq u_{Alice}(x, \sigma_{-Alice}(\omega))$, for all $x \in S_{Alice}$. It follows that *Alice* is rational at every world $\omega' \in \mathcal{I}_{Alice}(\omega)$. By definition of $\mathcal{I}_{Alice}$, since each cell contains a world ω such that $(\sigma_{Alice}(\omega), \sigma_{Bob}(\omega)) = (s^{2n}_{Alice}, s^{2n}_{Bob})$ for some $n \geq 0$, it follows that $R_{Alice} = \Omega$. Second, *Bob* is shown not to be rational at every possible world. In fact, his strategies $\sigma_{Bob}(\alpha_0)$, $\sigma_{Bob}(\beta_0)$, $\sigma_{Bob}(\gamma_0)$ and $\sigma_{Bob}(\delta_0)$ all equal 1, which in turn is strictly dominated by any $y \in (0, 1)$, thus $\alpha_0, \beta_0\gamma_0, \delta_0 \notin R_{Bob}$. Analogous reasoning as for *Alice* allows to conclude that *Bob* is rational at all remaining worlds. Therefore, $R_{Bob} = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$. Finally, *Claire* and *Donald* are rational at every possible world. Indeed, observe that *Claire* is rational at α, since $\alpha \in \mathcal{I}_{Claire}(\alpha)$ and $u_{Claire}(\sigma(\alpha)) \geq u_{Claire}(D, \sigma_{-Claire}(\alpha))$, where D is her only alternative strategy. As $\beta \in \mathcal{I}_{\text{Claire}}(\alpha)$, it follows that Claire is also rational at β. Similar arguments hold for *Claire*'s rationality at worlds γ and δ. Analogously, *Claire* is rational at all other possible worlds α_n, β_n, γ_n and δ_n, for all $n \geq 0$. Donald's rationality at each world is obtained in the same manner. Therefore, $R_{Claire} = R_{Donald} = \Omega$ and the event of all players being rational is given by $R = \bigcap_{i \in I} R_i = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$.

It follows that $K(R) = \bigcap_{i \in I} K_i(R) = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1\}$, and by induction $K^m(R) = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_m, \beta_m, \gamma_m, \delta_m\}$ for all $m > 0$. Therefore, $\text{CK}(R) = \bigcap_{m > 0} K^m(R) = \{\alpha, \beta, \gamma, \delta\}$. Besides, suppose the event space $\mathcal{P}(\Omega)$ to be equipped with the topology $\mathcal{T} = \{O \subseteq \mathcal{P}(\Omega) : \{\alpha\} \notin O\} \cup \{\mathcal{P}(\Omega)\}$.

Then, the only $\mathcal{T}$-open neighbourhood of the event $\{\alpha\}$ is $\mathcal{P}(\Omega)$, and all terms of the sequence $(K^m(R))_{m>0}$ are contained in $\mathcal{P}(\Omega)$. Thus $(K^m(R))_{m>0}$ converges to $\{\alpha\}$. Moreover, any singleton $\{F\} \neq \{\{\alpha\}\}$ is $\mathcal{T}$-open, and, since $K^{m+1}(R) \subsetneq K^m(R)$ for all $m > 0$, the strictly shrinking sequence $(K^m(R))_{m>0}$ will never remain in the open neighbourhood $\{F\}$ of F from some index onwards. Hence, $(K^m(R))_{m>0}$ does not converge to any such event F. Therefore the limit $(K^m(R))_{m>0}$ is unique, and $\mathrm{LK}(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = \{\alpha\}$.

Farther, $\sigma(\mathrm{CK}(R)) = \{\sigma(\alpha), \sigma(\beta), \sigma(\gamma), \sigma(\delta)\} = \{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\} = \mathrm{ISD}^\Gamma$, while $\sigma(\mathrm{LK}(R)) = \{\sigma(\alpha)\} = \{(\frac{1}{3}, \frac{1}{3}, U, L)\} = (\mathrm{ISD}+\mathrm{WD})^\Gamma$. Hence, the solution in accordance with $\mathrm{LK}(R)$ is a strict refinement of the solution induced by $\mathrm{CK}(R)$. ■

The preceding example describes a particular epistemic-topological epistemic model of a given game such that limit knowledge of rationality is a strict refinement of common knowledge of rationality in terms of solution concepts.

More generally, we now show that for any game and epistemic Aumann model of it such that the sequence of iterated mutual knowledge of rationality is strictly shrinking, every solution concept can be epistemic-topologically characterized by limit knowledge of rationality.

Theorem 1 *Let Γ be a game, $\mathcal{A}^\Gamma$ be an epistemic Aumann model of it such that the sequence $(K^m(R))_{m>0}$ is strictly shrinking, and $\mathcal{SC}$ be some solution concept. Then, there exists a topology on $\mathcal{P}(\Omega)$ such that $\sigma(LK(R)) \subseteq \mathcal{SC}^\Gamma$.*

Proof Consider the event $F = \sigma^{-1}(\mathcal{SC}^\Gamma) = \{\omega \in \Omega : \sigma(\omega) \in \mathcal{SC}^\Gamma\}$. Then $\sigma(F) \subseteq \mathcal{SC}^\Gamma$. Now, suppose the event space $\mathcal{P}(\Omega)$ to be equipped with the topology $\mathcal{T} = \{O \subseteq \mathcal{P}(\Omega) : F \notin O\} \cup \{\mathcal{P}(\Omega)\}$. By definition of $\mathcal{T}$, the only $\mathcal{T}$-open neighbourhood of F is $\mathcal{P}(\Omega)$, and thus the sequence $(K^m(R))_{m>0}$ converges to F. Besides, for every event $E \neq F$, the singleton $\{E\}$ is open, and, since satisfying the strictly shrinking condition, $(K^m(R))_{m>0}$ will never remain in the $\mathcal{T}$-open neighbourhood $\{E\}$ of E from some index onwards. Consequently, $(K^m(R))_{m>0}$ does not converge to E. It follows that the limit of $(K^m(R))_{m>0}$ is unique, and $\mathrm{LK}(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = F$. Therefore, $\sigma(\mathrm{LK}(R)) = \sigma(F) \subseteq \mathcal{SC}^\Gamma$. ■

As a matter of fact, limit knowledge of rationality can provide an epistemic-topological foundation for any game-theoretic solution concept. This universal characterization capability is enabled by choosing a tailored topology on the event space such that the epistemic-topological hypothesis gives a foundation for the desired solution concept. However, particular attention could be drawn to topologies on the event space that are plausible, such as topologies revealing some kind of underlying agent perceptions or reasoning patterns, as well as natural extension of implicit topologies on the state space.

In general, analogous to the epistemic program in game theory, which provides epistemic characterizations for solution concepts, an epistemic-topological approach to game theory is capable of epistemic-topologically characterizing solution concepts. Moreover, new solution concepts might be discovered by specific epistemic-topological assumptions.

5 Discussion

5.1 Topological Aumann structures

Aumann structures provide an abstract framework, in which the reasoning of agents about events on the basis of epistemic assumptions can be formalized. However, amending the epistemic framework by a topological dimension provides additional structure, enabling models of richer agent perceptions of the event and state spaces, as well as models of ample agent reasoning patterns that do not only depend on mere epistemic but also on topological features of the underlying interactive situation.

For instance, the event *It is cloudy in London* seems to be closer to the event *It is raining in London* than the event *It is sunny in London*. Now, agents may make identical decisions only being informed of the truth of some event within a class of *close* events. Indeed, Alice might decide to stay at home not only in the case of it raining outside, but also in the case of events perceived by her to be similar such as it being cloudy outside.

Actually, we envision the construction of a general topological framework—*topological Aumann structures*—for set-based interactive epistemology which comprises topologies on the state and event spaces.

Definition 5 A *topological Aumann structure* is a tuple $\mathcal{A}_{\mathcal{T}} = (\mathcal{A}, \mathcal{T}^{\Omega}, \mathcal{T}^{\mathcal{P}(\Omega)})$, where $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I})$ is an Aumann structure, $\mathcal{T}^{\Omega}$ is a topology on the state space Ω, and $\mathcal{T}^{\mathcal{P}(\Omega)}$ is a topology on the event space $\mathcal{P}(\Omega)$.

Accordingly, a topological component is added to the epistemic analysis of interactive situations. It thus becomes possible to model closeness of possible worlds as well as closeness of events.

Farther, it seems natural to require the topologies on the event space to depend on the topologies on the state space, and to thus restrict the class of topologies admissible on the event space. Indeed, topologies on spaces of subsets of a given topological space X are typically defined in terms of the topology of X, such as the Hausdorff or Vietories topologies. For instance, topologies on the event space could be given by these extensions of the topological framework in line with the usual measure-theoretic structure the state space.

Note that our topological approach to study the sequence of iterated mutual knowledge, compare it with common knowledge, and to define the new epistemic operator limit knowledge, serves as a first step towards such an topologically enriched epistemic framework.

5.2 Epistemically plausible topologies

Topologies representing epistemic features of a given underlying interactive situation or revealing particular agent perception patterns of the event or state spaces can be studied within the framework of topological Aumann structures.

For example, consider the partition topology on the state space generated by the basis $\mathcal{B} = \{O \subseteq \Omega : O = \bigcap_{i \in I} \mathcal{I}_i(\omega) \text{ for some } \omega \in \Omega\}$. Accordingly, every basic open set can thus be written as an intersection of the agents' possibility sets and interpreted as some kind of bundled, refined information of the agents. Notably, the partition topology represents the indistinguishability of worlds by all agents in topological terms. More precisely, any two possible worlds ω and ω' are indistinguishable by all agents if and only if ω and ω' are not separable, i.e., there do not exit two disjoint open sets O and O' such that $\omega \in O$ and $\omega' \in O'$. Equivalently, two possible worlds are distinguishable by some agents if and only if the two worlds are Hausdorff-separable. Hence, the partition topology precisely reflects informational indistinguishability in terms of closeness.

A further example of a topology representing epistemic features of the interactive situation is based on the notion of common truism. An event $T \subseteq \Omega$ is called common truism if and only if $\mathrm{CK}(T) = T$. Intuitively, a common truism event is directly commonly known, i.e., it cannot occur without being commonly known, and can hence be understood as a reliable piece of information that all agents receive in public announcement or joined observation type situations. In fact, Binmore aand Brandenburger (1990) already remark that the set of all common truisms form a topology. Notably, the common truism topology on the state space exhibits the property that two possible worlds ω and ω' are separable if and only if there exist two disjoint common truisms T and T' such that $\omega \in T$ and $\omega' \in T'$. Thus, worlds are separated by two different pieces of mutually exclusive self-evident information that considerably distinguish them in the sense that there exists no world whatsoever at which both pieces of information simultaneously hold.

5.3 Epistemic-topological characterizations of solution concepts

According to Theorem 1, limit knowledge of rationality can provide an epistemic-topological foundation for any game-theoretic solution concept. Therefore, an epistemic-topological approach to game theory can epistemic-topologically characterize solution concepts via limit knowledge of rationality. Relevant topological reasoning patterns of players in accordance with some given solution concept could thus be unveiled. Alternatively, solution concepts in accordance with limit knowledge of rationality could be derived, for some given topological assumptions. It might be of particular interest to explore the game-theoretic consequences of epistemically-plausible topologies being defined on the basis of underlying perception or reasoning patterns of the event space by the players, potentially revealing interesting new solution concepts.

As an example, a plausible epistemic-topological foundation for the solution concept k-times strict dominance SD^k is given now. Suppose a game in normal form Γ and some epistemic model $\mathcal{A}^\Gamma$ of it such that the sequence $(K^m(R))_{m>0}$ is strictly shrinking. Consider the topology $\mathcal{T}$ on $\mathcal{P}(\Omega)$ induced by the subbase

$$\begin{aligned}&\big\{\{K^m(R) : m > 0\}, \mathcal{P}(\Omega) \setminus \{K^m(R) : m > 0\}\big\}\\ &\quad\cup\ \big\{\{K^m(R)\} : m < k-1\big\}\\ &\quad\cup\ \Big\{\{K^k(R), K^{k+1}(R), \dots, K^n(R)\} : n > k-1\Big\}.\end{aligned}$$

This topology can be argued to be epistemically plausible in the sense that it satisfies the following four properties.

1. If E is a term of the sequence $(K^m(R))_{m>0}$ and F is not (or vice versa), then E and F are T_2-separable.[3]
2. If E and F are two distinct terms of $(K^m(R))_{m>0}$ of index strictly smaller than $k-1$, then E and F are T_2-separable.
3. If E and F are two distinct terms of $(K^m(R))_{m>0}$ of index strictly larger than $k-1$, then E and F are T_0-separable.[4]
4. If $E = K^{k-1}(R)$ and F is another term of $(K^m(R))_{m>0}$ (or vice versa), then E and F are T_0-separable.

These properties reflect a particular perception of the event space, where the agents' topological distinction between the first $(k-2)$-order knowledge of rationality is stronger than between the remaining higher-order mutual knowledge. By definition of $\mathcal{T}$, it follows that $\text{LK}(R) = K^{k-1}(R)$ and therefore the proof of Proposition 1 ensures that $\sigma(\text{LK}(R)) \subseteq \text{SD}^k$. In this sense, $\mathcal{T}$ provides a plausible epistemic-topological foundation for the solution concept SD^k.

5.4 Counterfactuals

Closeness of possible worlds can be quantified within the framework of topological Aumann structures. Indeed, metrizable or pseudo-metrizable topologies on the state space induce a distance measure for possible worlds. For instance, consider the pseudo-metric $d : \Omega \times \Omega \to \mathbb{R}$, defined by $d(\omega, \omega') = k$ for all $\omega, \omega' \in \Omega$, where k equals the number of agents being able to distinguish between ω and ω'. This pseudo-metric provides a closeness measure for possible worlds and induces a topological Aumann structure $\mathcal{A}_{\mathcal{T}}$ equipped with the partition topology on the state space.

Now, a notion of closeness of worlds is typically needed for theories of counterfactual reasoning. Hence, the enriched framework of topological Aumann structures could be used to model counterfactual knowledge and reasoning in set-based interactive epistemology. For instance, if some event E does not hold at the actual world ω, the reasoning of an agent i may depend on whether E is nevertheless close to what he actually considers possible, i.e., on whether all worlds contained in E are closer to every world $\omega' \in \mathcal{I}_i(\omega)$ than all worlds contained in $\Omega \setminus (E \cup \mathcal{I}_i(\omega))$.

6 Conclusions

The standard game-theoretic concept of common knowledge has been shown to differ from the topological limit of the sequence of iterated mutual knowledge, and, on the basis of this result, the new epistemic operator limit knowledge introduced. The

[3] Given a topological space $(X, \mathcal{T})$ two points $a, b \in X$ are called T_2-separable if there exist two disjoint $\mathcal{T}$-neighbourhoods N_a and N_b of a and b, respectively.

[4] Given a topological space $(X, \mathcal{T})$ two points in X are called T_0-separable if there exists an $\mathcal{T}$-open set containing precisely one of the two points.

specific hypothesis limit knowledge of rationality turns out to be capable of epistemic-topological characterizations of solution concepts in games. Hence, analogous to the epistemic program in game theory that attempts to provide epistemic foundations for solution concepts, our epistemic-topological approach to game theory could generate epistemic-topological foundations for solution concepts via limit knowledge of rationality.

More generally, the notion of topological Aumann structure envisioned here provides a topological component to the epistemic analysis of interactive situations. Such an extended epistemic-topological framework thus enables models of richer agent perceptions of the event and state spaces, as well as models of ample agent reasoning patterns that do not only depend on mere epistemic but also on topological features of the underlying interactive situation.

Finally, the topological approach to set-based interactive epistemology initiated here can be extended in various directions. Indeed, epistemic-topological reasoning patterns of players underlying solution concepts or leading to new solution concepts may be unveiled. Besides, based on the notion of closeness of possible worlds, furnished by topological properties of the state space, a theory of counterfactuals could be developed. Farther, Aumann's (1976) seminal impossibility result on agreeing to disagree can be reconsidered with limit knowledge instead of common knowledge of the agents' posteriors beliefs.

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