The Algebraic Counterpart of the Wagner Hierarchy

Jérémie Cabessa and Jacques Duparc

Université de Lausanne,
Faculty of Business and Economics HEC, Institute of Information Systems ISI,
CH-1015 Lausanne, Switzerland
Jeremie.Cabessa@unil.ch, Jacques.Duparc@unil.ch

Abstract. The algebraic study of formal languages shows that ω-rational languages are exactly the sets recognizable by finite ω-semigroups. Within this framework, we provide a construction of the algebraic counterpart of the Wagner hierarchy. We adopt a hierarchical game approach, by translating the Wadge theory from the ω-rational language to the ω-semigroup context.

More precisely, we first define a reduction relation on finite pointed ω-semigroups by means of a Wadge-like infinite two-player game. The collection of these algebraic structures ordered by this reduction is then proven to be isomorphic to the Wagner hierarchy, namely a decidable and well-founded partial ordering of width 2 and height ω^ω.

Keywords: ω-automata, ω-rational languages, ω-semigroups, infinite games, Wadge game, Wadge hierarchy, Wagner hierarchy.

1 Introduction

This paper stands at the crossroads of two mathematical fields, namely the algebraic theory of ω-automata, and hierarchical games, in descriptive set theory.

The basic interest of the algebraic approach to automata theory consists in the equivalence between Büchi automata and finite ω-semigroups [12] – an extension of the concept of a semigroup. These mathematical objects indeed satisfy several relevant properties. Firstly, given a finite Büchi automaton, one can effectively compute a finite ω-semigroup recognizing the same ω-language, and vice versa. Secondly, among all finite ω-semigroups recognizing a given ω-language, there exists a minimal one – called the syntactic ω-semigroup –, whereas there is no convincing notion of Büchi (or Muller) minimal automaton. Thirdly, finite ω-semigroups provide powerful characteristics towards the classification of ω-rational languages; for instance, an ω-language is first-order definable if and only if it is recognized by an aperiodic ω-semigroup [7,10,18], a generalization to infinite words of Schützenberger, and McNaughton's and Papert famous results [9,16]. Even some topological properties (being open, closed, clopen, $\mathbf{\Sigma^0_2}$, $\mathbf{\Pi^0_2}$, $\mathbf{\Delta^0_2}$) can be characterized by algebraic properties on ω-semigroups (see [12,14]).

Hierarchical games, for their part, aim to classify subsets of topological spaces. In particular, the Wadge hierarchy [19] (defined via the Wadge games) appeared

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to be specially interesting to computer scientists, for it shed a light on the study of classifying ω-rational languages. The famous Wagner hierarchy [20], known as the most refined classification of ω-rational languages, was proven to be precisely the restriction of the Wadge hierarchy to these ω-languages.

However, Wagner's original construction relies on a graph-theoretic analysis of Muller automata, away from the set theoretical and the algebraic frameworks. Olivier Carton and Dominique Perrin [2,3,4] investigated the algebraic reformulation of the Wagner hierarchy, a work carried on by Jacques Duparc and Mariane Riss [6]. But this new approach is not yet entirely satisfactory, for it fails to define precisely the algebraic counterpart of the Wadge (or Wagner) preorder on finite ω-semigroups.

Our paper fill this gap. We define a reduction relation on subsets of finite ω-semigroups by means of an infinite game, without any direct reference to the Wagner hierarchy. We then show that the resulting algebraic hierarchy is isomorphic to the Wagner hierarchy, and in this sense corresponds to the algebraic counterpart of the Wagner hierarchy. In particular, this classification is a refinement of the hierarchies of chains and superchains introduced in [2,4]. We finally prove that this algebraic hierarchy is also decidable.

2 Preliminaries

2.1 ω-Languages

Given a finite set A, called the *alphabet*, then A^*, A^+, A^ω, and A^∞ denote respectively the sets of finite words, nonempty finite words, infinite words, and finite or infinite words, all of them over the alphabet A. Given a finite word u and a finite or infinite word v, then uv denotes the concatenation of u and v. Given $X \subseteq A^*$ and $Y \subseteq A^\infty$, the concatenation of X and Y is denoted by XY.

We refer to [12, p.15] for the definition of *ω-rational languages*. We recall that ω-rational languages are exactly the ones recognized by finite Büchi, or equivalently, by finite Muller automata [12].

For any set A, the set A^ω can be equipped with the product topology of the discrete topology on A. The class of *Borel* subsets of A^ω is the smallest class containing the open sets, and closed under countable union and complementation.

2.2 ω-Semigroups

The notion of an ω-semigroup was first introduced by Pin as a generalization of semigroups [11,13]. In the case of finite structures, these objects represent a convincing algebraic counterpart to automata reading infinite words: given any finite Büchi automaton, one can build a finite ω-semigroup recognizing (in an algebraic sense) the same language, and conversely, given any finite ω-semigroup recognizing a certain language, one can build a finite Büchi automaton recognizing the same language.

Definition 1 (see [12, p. 92]). *An* ω-semigroup *is an algebra consisting of two components, $S = (S_+, S_\omega)$, and equipped with the following operations:*

- *a binary operation on S_+, denoted multiplicatively, such that S_+ equipped with this operation is a semigroup;*
- *a mapping $S_+ \times S_\omega \longrightarrow S_\omega$, called mixed product, which associates with each pair $(s,t) \in S_+ \times S_\omega$ an element of S_ω, denoted by st, and such that for every $s,t \in S_+$ and for every $u \in S_\omega$, then $s(tu) = (st)u$;*
- *a surjective mapping $\pi_S : S_+^\omega \longrightarrow S_\omega$, called infinite product, such that: for every strictly increasing sequence of integers $(k_n)_{n>0}$, for every sequence $(s_n)_{n\geq 0} \in S_+^\omega$, and for every $s \in S_+$, then*

$$\pi_S(s_0 s_1 \cdots s_{k_1-1}, s_{k_1} \cdots s_{k_2-1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$
$$s\pi_S(s_0, s_1, s_2, \ldots) = \pi_S(s, s_0, s_1, s_2, \ldots).$$

Intuitively, an ω-semigroup is a semigroup equipped with a suitable infinite product. The conditions on the infinite product ensure that one can replace the notation $\pi_S(s_0, s_1, s_2, \ldots)$ by the notation $s_0 s_1 s_2 \cdots$ without ambiguity. Since an ω-semigroup is a pair (S_+, S_ω), it is convenient to call $+$*-subsets* and ω*-subsets* the subsets of S_+ and S_ω, respectively.

Given two ω-semigroups $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$, a *morphism of* ω*-semigroups* from S into T is a pair $\varphi = (\varphi_+, \varphi_\omega)$, where $\varphi_+ : S_+ \longrightarrow T_+$ is a morphism of semigroups, and $\varphi_\omega : S_\omega \longrightarrow T_\omega$ is a mapping canonically induced by φ_+ in order to preserve the infinite product, that is, for every sequence $(s_n)_{n\geq 0}$ of elements of S_+, one has $\varphi_\omega\big(\pi_S(s_0, s_1, s_2, \ldots)\big) = \pi_T\big(\varphi_+(s_0), \varphi_+(s_1), \varphi_+(s_2), \ldots\big)$.

An ω-semigroup S is an ω*-subsemigroup* of T if there exists an injective morphism of ω-semigroups from S into T. An ω-semigroup S is a *quotient* of T if there exists a surjective morphism of ω-semigroups from T onto S. An ω-semigroup S *divides* T if S is quotient of an ω-subsemigroup of T.

The notion of pointed ω-semigroup can be adapted from the notion of pointed semigroup introduced by Sakarovitch [15]. In this paper, a *pointed* ω*-semigroup* denotes a pair (S, X), where S is an ω-semigroup and X is an ω-subset of S. A mapping $\varphi : (S, X) \longrightarrow (T, Y)$ is a morphism of pointed ω-semigroups if $\varphi : S \longrightarrow T$ is a morphism of ω-semigroups such that $\varphi^{-1}(Y) = X$. The notions of ω-subsemigroups, quotient, and division can then be easily adapted in this pointed context.

Example 1. Let A be a finite set. The ω-semigroup $A^\infty = (A^+, A^\omega)$ equipped with the usual concatenation is the *free* ω*-semigroup* over the alphabet A [2]. In addition, if $S = (S_+, S_\omega)$ is an ω-semigroup with S_+ being finite, the morphism of ω-semigroups $\varphi : S_+^\infty \longrightarrow S$ naturally induced by the identity over S_+ is called the *canonical morphism* associated with S.

In this paper, we strictly focus on *finite* ω-semigroups, i.e. those whose first component is finite. It is proven in [12] that the infinite product π_S of a finite ω-semigroup S is completely determined be the mixed products of the form $x\pi_S(s, s, s, \ldots)$ (denoted xs^ω). We use this property in the next example.

Example 2. The pair $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ equipped with the usual multiplication over $\{0,1\}$ and with the infinite product defined by the relations $00^\omega = 10^\omega = 0^\omega$ and $01^\omega = 11^\omega = 1^\omega$ is an ω-semigroup.

Wilke was the first to give the appropriate algebraic counterpart to finite automata reading infinite words [21]. In addition, he established that the ω-languages recognized by finite ω-semigroups are exactly the ones recognized by Büchi automata, a proof that can be found in [21] or [12].

Definition 2. *Let S and T be two ω-semigroups. One says that a surjective morphism of ω-semigroups $\varphi : S \longrightarrow T$ recognizes a subset X of S if there exists a subset Y of T such that $\varphi^{-1}(Y) = X$. By extension, one also says that the ω-semigroup T recognizes X.*

Proposition 1 (Wilke). *An ω-language is recognizable by a finite ω-semigroup if and only if it is ω-rational.*

Example 3. Let $A = \{a, b\}$, let S be the ω-semigroup given in Example 2, and let $\varphi : A^\infty \longrightarrow S$ be the morphism defined by $\varphi(a) = 0$ and $\varphi(b) = 1$. Then $\varphi^{-1}(0^\omega) = (A^*a)^\omega$ and $\varphi^{-1}(1^\omega) = A^*b^\omega$, and therefore these two languages are ω-rational.

A *congruence of an ω-semigroup* $S = (S_+, S_\omega)$ is a pair $(\sim_+, \sim_\omega)$, where $\sim_+$ is a semigroup congruence on S_+, $\sim_\omega$ is an equivalence relation on S_ω, and these relations are stable for the infinite and the mixed products (see [12]). The quotient set $S/\!\sim\, = (S/\!\sim_+, S/\!\sim_\omega)$ is naturally equipped with a structure of ω-semigroup. If $(\sim_i)_{i\in I}$ is a family of congruences on an ω-semigroup, then the congruence $\sim$, defined by $s \sim t$ if and only if $s \sim_i t$, for all $i \in I$, is called the lower bound of the family $(\sim_i)_{i\in I}$. The upper bound of the family $(\sim_i)_{i\in I}$ is then the lower bound of the congruences that are coarser than all the $\sim_i$.

Given a subset X of an ω-semigroup S, the *syntactic congruence* of X, denoted by $\sim_X$, is the upper bound of the family of congruences whose associated quotient morphisms recognize X, if this upper bound still recognizes X, and is undefined otherwise. Whenever defined, the quotient $S(X) = S/\!\sim_X$ is called the *syntactic ω-semigroup* of X, the surjective morphism $\mu : S \longrightarrow S(X)$ is the *syntactic morphism* of X, the set $\mu(X)$ is the *syntactic image* of X, and one has the property $\mu^{-1}(\mu(X)) = X$. The pointed ω-semigroup $(S(X), \mu(X))$ will be denoted by $Synt(X)$. One can prove that the syntactic ω-semigroup of an ω-rational language is always defined, and is the unique (up to isomorphism) and minimal (for the division) pointed ω-semigroup recognizing this language [12].

Example 4. Let $K = (A^*a)^\omega$ be an ω-language over the alphabet $A = \{a, b\}$. The morphism $\varphi : A^\infty \longrightarrow S$ given in Example 3 is the syntactic morphism of K. The ω-subset $X = \{0^\omega\}$ of S is the syntactic image of K.

Finally, a pointed ω-semigroup (S, X) will be called *Borel* if the preimage $\pi_S^{-1}(X)$ is a Borel subset of S_+^ω (where S_+^ω is equipped with the product topology of the discrete topology on S_+). Notice that every finite pointed ω-semigroup is Borel, since by Proposition 1, its preimage by the infinite product is ω-rational, hence Borel (more precisely boolean combination of $\mathbf{\Sigma_2^0}$) [12].

3 The Wadge and the Wagner Hierarchies

Let A and B be two alphabets, and let $X \subseteq A^\omega$ and $Y \subseteq B^\omega$. The *Wadge game* $\mathbb{W}((A,X),(B,Y))$ [19] is a two-player infinite game with perfect information, where Player I is in charge of the subset X and Player II is in charge of the subset Y. Players I and II alternately play letters from the alphabets A and B, respectively. Player I begins. Player II is allowed to skip her turn – formally denoted by the symbol "$-$" – provided she plays infinitely many letters, whereas Player I is not allowed to do so. After ω turns each, players I and II respectively produced two infinite words $\alpha \in A^\omega$ and $\beta \in B^\omega$. Player II wins $\mathbb{W}((A,X),(B,Y))$ if and only if $(\alpha \in X \Leftrightarrow \beta \in Y)$. From this point onward, the Wadge game $\mathbb{W}((A,X),(B,Y))$ will be denoted $\mathbb{W}(X,Y)$ and the alphabets involved will always be clear from the context.

Along the play, the finite sequence of all previous moves of a given player is called the *current position* of this player. A *strategy* for Player I is a mapping from $(B \cup \{-\})^*$ into A. A *strategy* for Player II is a mapping from A^+ into $B \cup \{-\}$. A strategy is *winning* if the player following it must necessarily win, no matter what his opponent plays.

The *Wadge reduction* is defined via the Wadge game as follows: a set X is said to be *Wadge reducible* to Y, denoted by $X \leq_W Y$, if and only if Player II has a winning strategy in $\mathbb{W}(X,Y)$. One then sets $X \equiv_W Y$ if and only if both $X \leq_W Y$ and $Y \leq_W X$, and also $X <_W Y$ if and only if $X \leq_W Y$ and $X \not\equiv_W Y$. The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. A set X is called *self-dual* if $X \equiv_W X^c$, and *non-self-dual* if $X \not\equiv_W X^c$. One can show [19] that the Wadge reduction coincides with the continuous reduction, that is $X \leq_W Y$ if and only if $f^{-1}(Y) = X$, for some continuous function $f : A^\omega \longrightarrow B^\omega$.

The Wadge hierarchy consists of the collection of all ω-languages ordered by the Wadge reduction, and *the Borel Wadge hierarchy* is the restriction of the Wadge hierarchy to Borel ω-languages. Martin's Borel determinacy [8] easily implies Borel Wadge determinacy, that is, whenever X and Y are Borel sets, then one of the two players has a winning strategy in $\mathbb{W}(X,Y)$. As a corollary, one can prove that, up to complementation and Wadge equivalence, the Borel Wadge hierarchy is a well ordering. Therefore, there exist a unique ordinal, called the *height* of the Borel Wadge hierarchy, and a mapping d_W from the Borel Wadge hierarchy onto its height, called the *Wadge degree*, such that $d_W(X) < d_W(Y)$ if and only if $X <_W Y$, and $d_W(X) = d_W(Y)$ if and only if either $X \equiv_W Y$ or $X \equiv_W Y^c$, for every Borel ω-languages X and Y. The Borel Wadge hierarchy actually consists of an alternating succession of non-self-dual and self-dual sets with non-self-dual pairs at each limit level (as soon as finite alphabets are considered) [5,19].

The Wagner hierarchy is precisely the restriction of the Wadge hierarchy to ω-rational languages, and hence corresponds to the most refined classification of such languages [6,12,20]. This hierarchy has a height of ω^ω, and it is decidable. The Wagner degree of an ω-rational language can indeed be computed by analyzing the graph of a Muller automaton accepting this language [20].

Selivanov gave a complete set theoretical description of the Wagner hierarchy in terms of boolean expressions [17], and Carton, Perrin, Duparc, and Riss studied some algebraic properties of this hierarchy [2,4,6]. In this context, the present work provides a complete construction of the algebraic counterpart of the Wagner hierarchy.

4 The $\mathbb{SG}$-Hierarchy

We define a reduction relation on pointed ω-semigroups by means of an infinite two-player game. This reduction induces a hierarchy of pointed ω-semigroups. Many results of the Wadge theory [19] also apply in this framework, and provide a detailed description of this algebraic hierarchy.

Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two ω-subsets. The game $\mathbb{SG}((S, X), (T, Y))$ is an infinite two-player game with perfect information, where Player I is in charge of X, Player II is in charge of Y, and players I and II alternately play elements of S_+ and $T_+ \cup \{-\}$, respectively. Player I begins. Unlike Player I, Player II is allowed to skip her turn – denoted by the symbol "$-$" –, provided she plays infinitely many moves. After ω turns each, players I and II produced respectively two infinite sequences $(s_0, s_1, \ldots) \in S_+^\omega$ and $(t_0, t_1, \ldots) \in T_+^\omega$. Player II wins $\mathbb{SG}((S, X), (T, Y))$ if and only if $\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y$. From this point onward, the game $\mathbb{SG}((S, X), (T, Y))$ will be denoted by $\mathbb{SG}(X, Y)$ and the ω-semigroups involved will always be known from the context. A play in this game is illustrated below.

$$
\begin{array}{lccccc}
(X)\ \mathrm{I}\ : & s_0 & & s_1 \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (s_0, s_1, s_2, \ldots) \\
 & & \searrow \quad \nearrow & & & \\
(Y)\ \mathrm{II}\ : & & t_0 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (t_0, t_1, t_2, \ldots)
\end{array}
$$

We now say that X is $\mathbb{SG}$-*reducible* to Y, denoted by $X \leq_{SG} Y$, if and only if Player II has a winning strategy in $\mathbb{SG}(X, Y)$. We then naturally set $X \equiv_{SG} Y$ if and only if both $X \leq_{SG} Y$ and $Y \leq_{SG} X$, and also $X <_{SG} Y$ if and only if $X \leq_{SG} Y$ and $X \not\equiv_{SG} Y$. The relation $\leq_{SG}$ is reflexive and transitive, and $\equiv_{SG}$ is an equivalence relation.

Notice that if (S, X) and (T, Y) are two pointed ω-semigroups, a given player has a winning strategy in the game $\mathbb{SG}(X, Y)$ if and only if this same player also has one in the Wadge game $\mathbb{W}(\pi_S^{-1}(X), \pi_T^{-1}(Y))$. Therefore Borel Wadge determinacy implies the determinacy of $\mathbb{SG}$-games involving Borel pointed ω-semigroups.

The collection of Borel pointed ω-semigroups ordered by the $\leq_{SG}$-relation is called *the* $\mathbb{SG}$-*hierarchy*, in order to underline the semigroup approach. Notice that the restriction of the $\mathbb{SG}$-hierarchy to Borel pointed free ω-semigroups is exactly the Borel Wadge hierarchy. When restricted to finite pointed ω-semigroups, this hierarchy will be called *the* $\mathbb{FSG}$*-hierarchy*, in order to underline the finiteness of the ω-semigroups involved. As corollaries of the determinacy of Borel $\mathbb{SG}$-games, a straightforward generalization in this context of Martin and Wadge's results [8,19] shows that, up to complementation and $\leq_{SG}$-equivalence, the $\mathbb{SG}$-hierarchy is a well ordering. Therefore, there exist again a unique ordinal, called the *height* of the $\mathbb{SG}$-hierarchy, and a mapping d_{SG} from the $\mathbb{SG}$-hierarchy onto its height, called the $\mathbb{SG}$-*degree*, such that $d_{SG}(X) < d_{SG}(Y)$ if and only if $X <_{SG} Y$, and $d_{SG}(X) = d_{SG}(Y)$ if and only if either $X \equiv_{SG} Y$ or $X \equiv_{SG} Y^c$, for every Borel ω-subsets X and Y. It directly follows from the Wadge analysis that the $\mathbb{SG}$-hierarchy has the same familiar "scaling shape" as the Borel or Wadge hierarchies: an increasing sequence of non-self-dual sets with self-dual sets in between, as illustrated in Figure 1, where circles represent the $\equiv_{SG}$-equivalence classes of pointed ω-semigroups, and arrows stand for the $<_{SG}$-relation.

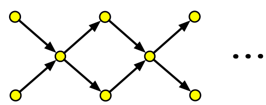

Fig. 1. The $\mathbb{SG}$-hierarchy

5 The $\mathbb{FSG}$ and the Wagner Hierarchies

This section shows that the $\mathbb{FSG}$-hierarchy is precisely the algebraic counterpart of the Wagner hierarchy. Consequently, this algebraic hierarchy has a height of ω^ω, and it is decidable.

Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and let $\varphi : A^\infty \longrightarrow S$ be a surjective morphism of ω-semigroups, for some finite alphabet A. Then every ω-subset X of S_ω can be lifted on an ω-rational language $\varphi^{-1}(X)$ of A^ω. The next proposition proves that this lifting induces an embedding from the $\mathbb{FSG}$-hierarchy into the Wagner hierarchy.

Proposition 2. *Let (S, X) and (T, Y) be two finite pointed ω-semigroups, and let $\varphi : A^\infty \longrightarrow S$ and $\psi : B^\infty \longrightarrow T$ be two surjective morphisms of ω-semigroups, where A and B are finite alphabets. Then $X \leq_{SG} Y$ if and only if $\varphi^{-1}(X) \leq_W \psi^{-1}(Y)$.*

Proof (sketch). A complete proof can be found in [1, pp. 86–88]. For the first direction, a given winning strategy for Player II in $\mathbb{SG}(X, Y)$ induces via φ and ψ^{-1} a winning strategy for this same player in the game $\mathbb{W}\left(\varphi^{-1}(X), \psi^{-1}(Y)\right)$. Conversely, a given winning strategy for Player II in $\mathbb{W}\left(\varphi^{-1}(X), \psi^{-1}(Y)\right)$ also induces via φ^{-1} and ψ a winning strategy for this same player in $\mathbb{SG}(X, Y)$. $\square$

By the previous proposition, the Wadge reduction on ω-rational languages and the $\mathbb{SG}$-reduction on ω-subsets recognizing these languages coincide. The next corollary mentions that this property holds in particular for ω-rational languages and their syntactic images, meaning that the $\mathbb{SG}$-reduction is the appropriate algebraic counterpart of the Wagner reduction. As a direct consequence, the Wagner degree is a *syntactic invariant*: if two ω-rational languages have the same syntactic image, then they also have the same Wagner degree.

Corollary 1. *Let K and L be two ω-rational languages and $\mu(K)$ and $\nu(L)$ be their syntactic images.*

(1) *$K \leq_W L$ if and only if $\mu(K) \leq_{SG} \nu(L)$.*

(2) *If $Synt(K) = Synt(L)$, then $K \equiv_W L$.*

Proof. Since μ and ν are syntactic morphisms, one has $\mu^{-1}(\mu(K)) = K$ and $\nu^{-1}(\nu(L)) = L$. Proposition 2 leads to the conclusion. For (2), if $Synt(K) = Synt(L)$, then $\mu(K) = \nu(L)$, and (1) leads to the conclusion. $\square$

As another consequence, the $\mathbb{SG}$-degree of an ω-subset is invariant under surjective morphism, and in particular under syntactic morphism. Therefore, syntactic finite pointed ω-semigroups are minimal representatives of their $\leq_{SG}$-equivalence class.

Corollary 2. *Let $\mu : S \longrightarrow T$ be a surjective morphism of finite ω-semigroups, let $Y \subseteq T_\omega$, and let $X = \mu^{-1}(Y)$. Then $X \equiv_{SG} Y$.*

Proof. Let $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism of ω-semigroups associated with S, and let $\psi = \mu \circ \varphi : S_+^\infty \longrightarrow T$. The mapping ψ is a surjective morphism of ω-semigroups. It satisfies $\psi^{-1}(Y) = \varphi^{-1} \circ \mu^{-1}(Y) = \varphi^{-1}(X)$, thus in particular, $\varphi^{-1}(X) \equiv_W \psi^{-1}(Y)$. Proposition 2 then shows that $X \equiv_{SG} Y$. $\square$

Finally, the following theorem proves that the Wagner hierarchy and the $\mathbb{FSG}$-hierarchy are isomorphic. The required isomorphism is the mapping which associates every ω-rational language with its syntactic image. Therefore, the Wagner degree of an ω-rational language and the $\mathbb{SG}$-degree of its syntactic image are the same.

Theorem 1. *The Wagner hierarchy and the $\mathbb{FSG}$-hierarchy are isomorphic.*

Proof. Consider the mapping from the Wagner hierarchy into the $\mathbb{SG}$-hierarchy which associates every ω-rational language with its syntactic image. We prove that this mapping is an embedding. Let K and L be two ω-rational languages, and let $X = \mu(K)$ and $Y = \nu(L)$ be their syntactic images. Corollary 1 ensures that $K \leq_W L$ if and only if $X \leq_{SG} Y$. We now show that, up to $\equiv_{SG}$-equivalence, this mapping is onto. Let X be an ω-subset of a finite ω-semigroup $S = (S_+, S_\omega)$, let $\mu : S \longrightarrow S(X)$ be the syntactic morphism of X, and let $Y = \mu(X)$ be its syntactic image. Corollary 2 ensures that $X \equiv_{SG} Y$. Now, let also $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism associated with S_+, and let $L = \varphi^{-1}(X)$. Then the morphism of ω-semigroups $\psi = \mu \circ \varphi : S_+^\infty \longrightarrow S(X)$ is the syntactic morphism of L [12], and one has $\psi(L) = Y \equiv_{SG} X$. $\square$

As a corollary, we show that the $\mathbb{FSG}$-hierarchy is *decidable*: for every ω-subset X of the hierarchy, one can effectively compute the Cantor normal form of base ω of the ordinal $d_{SG}(X)$.

Corollary 3. *The $\mathbb{FSG}$-hierarchy has height ω^ω, and it is decidable.*

Proof. By the previous theorem, the $\mathbb{FSG}$-hierarchy and the Wagner hierarchy have the same height, namely ω^ω. In addition, given an ω-subset X of a finite ω-semigroup $S = (S_+, S_\omega)$, one can effectively compute the $\mathbb{SG}$-degree of X as follows. Let $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism associated with S_+, and let $L = \varphi^{-1}(X)$. Theorem 1 shows that the $\mathbb{SG}$-degree of X is equal to the Wagner degree of L. Furthermore, the Wagner degree of L can be effectively computed as follows. First, one can effectively compute an ω-rational expression describing $L = \varphi^{-1}(X)$ [12, Corollary 7.4, p. 110]. Next, one can shift from this rational expression to some finite Muller automaton recognizing L, see [12, Chapter I, sections 10.1, 10.3, and 10.4]. Finally, the Wagner degree of the ω-language recognized by a finite Muller automaton is effectively computable [20]. $\square$

Example 5. Consider the syntactic image (S, X) of the ω-language $K = (A^*a)^\omega$ given in example 4. We can prove that $d_{SG}((S, X)) = d_W(K) = \omega$.

6 Conclusion

This work is a first step towards the complete description of the algebraic counterpart of the Wagner hierarchy. Using a hierarchical game approach, we defined a reduction relation on finite pointed ω-semigroups which was proven to be the algebraic counterpart of the Wadge (or Wagner) preorder on ω-rational languages. As a direct consequence, the Wagner degree of ω-rational languages is a syntactic invariant. The resulting algebraic hierarchy is then isomorphic to the Wagner hierarchy, namely a decidable partial order of width 2 and height ω^ω. But the decidability procedure presented in Corollary 3 relies on Wagner's naming procedure over Muller automata, and in this sense withdraws from the purely algebraic context.

The natural extension of this work would be to fill this gap, and hence describe an algorithm computing the Wagner degree of any ω-rational set directly on its syntactic pointed ω-semigroup, without any reference to some underlying Muller automata. This study is the purpose of a forthcoming paper.

We can also hope to extend this work to more sophisticated ω-languages, like those recognized by deterministic counters, or even deterministic pushdown automata. This would obviously require to understand first the kind of infinite ω-semigroups corresponding to such machines.

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