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Papers of the conference "*Foundations of the Formal Sciences V*"
held in Bonn, November 26-29, 2004

An Infinite Game over ω-Semigroups

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Abstract. Jean-Éric Pin introduced the structure of an ω-semigroup in [PerPin04] as an algebraic counterpart to the concept of automaton reading infinite words. It has been well studied since, specially by Carton, Perrin [CarPer97] and [CarPer99], and Wilke. We introduce a reduction relation on subsets of ω-semigroups defined by way of an infinite two-player game. Both Wadge hierarchy and Wagner hierarchy become special cases of the hierarchy induced by this reduction relation. But on the other hand, set theoretical properties that occur naturally when studying these hierarchies, happen to have a decisive algebraic counterpart. A game theoretical characterization of basic algebraic concepts follows.

1 Introduction

This work comes from an interaction between classical game theory, and the algebra of automata theory, which rests on the following main facts. In case of finite words, a well-known correspondence between an automaton and a finite semigroup exists: from any finite automaton $\mathcal{A}$

Received: ...;
In revised version: ...;
Accepted by the editors: ....
2000 *Mathematics Subject Classification.* **PRIMARY** SECONDARY.

* This line is for acknowledgments.

recognizing a regular language L, one can build a finite semigroup $S_{\mathcal{A}}$ recognizing (in an algebraic way) the same language, and vice-versa [PerPin04]. Moreover, this correspondence generalizes in case of infinite words. Indeed, for that purpose, J.-É. Pin introduced the structure of ω-semigroup [PerPin04] as an algebraic counterpart to the concept of an automaton on infinite words. More precisely, he proved *the equivalence* between a finite Büchi automaton and a finite ω-semigroup.

This paper presents a game theoretical study of the structure of ω-semigroup, leading to an expected new foundation of the Wagner hierarchy, but also to promising general set theoretical, and algebraic results.

2 Preliminaries

We recall that a relation R is a *preorder* if it is reflexive, and transitive. It is a *partial order* if it is reflexive, transitive, and antisymmetric. And it is an *equivalence relation* if it is reflexive, transitive, and symmetric.

Given a set A (called the alphabet), we respectively denote by $\boldsymbol{A^*}$, $\boldsymbol{A^+}$, $\boldsymbol{A^\omega}$, the sets of finite words over A, non empty finite words over A, and infinite words over A. We set $\boldsymbol{A^\infty} = A^* \cup A^\omega$, and the empty word is denoted by $\boldsymbol{\epsilon}$. Given two words u and v (u finite), we write $\boldsymbol{uv}$ for the concatenation of u and v, $\boldsymbol{u \subseteq v}$ for "u is an initial segment of v", $\boldsymbol{v \upharpoonright n}$ for the restriction of v to its n first letters. Given $X \subseteq A^*$, and $Y \subseteq A^\infty$, we set: $\boldsymbol{XY} = \{xy : x \in X \wedge y \in Y\}$, $\boldsymbol{X^*} = \{x_1 \cdots x_n : n \geq 0 \wedge x_1, \ldots, x_n \in X\}$, $\boldsymbol{X^+} = \{x_1 \cdots x_n : n > 0 \wedge x_1, \ldots, x_n \in X\}$, and $\boldsymbol{X^\omega} = \{x_0 x_1 x_2 \cdots : \forall\, n \geq 0, x_n \in X\}$. The class of ω-*rational* subsets of A^∞ is the smallest class of subsets of A^∞ containing the finite subsets of A^∞, and closed under finite union, finite product, and both operations $X \to X^*$, and $X \to X^\omega$.

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative operation from $S \times S$ into S. A *morphism of semigroups* is a map ϕ from a semigroup S into a semigroup T such that $\forall\, s_1, s_2 \in S, \phi(s_1 s_2) = \phi(s_1)\phi(s_2)$ holds. A *monoid* is a set equipped with an associative operation, and an identity element. If S is a semigroup, S^1 denotes S if S is a monoid, and $S \cup \{\mathbf{1}\}$ otherwise (with the operation of S completed as follows: $\mathbf{1} \cdot s = s \cdot \mathbf{1} = s$, $\forall s \in S$). A *group* G is a monoid such that every element has an inverse, i.e. $\forall\, s \in G\ \exists\, s^{-1} \in G$ s.t. $s^{-1} \cdot s = s \cdot s^{-1} = \mathbf{1}$.

For any set A, the set A^ω is a topological space equipped with the product topology of the discrete topology on A. The basic open sets of

A^ω are of the form WA^ω, where $W \subseteq A^*$. Given a topological space E, the class of *Borel* subsets of E is the smallest class containing the open sets, and closed under countable union, and complementation. Let $F \subseteq 2^\omega$, F is a ***flip set*** [And03] iff $\forall\ x, y \in 2^\omega (\exists!\ k \in \omega\ (x(k) \neq y(k))) \rightarrow (x \in F \leftrightarrow y \notin F)$. We use the fact that a flip set cannot be Borel (as it doesn't satisfy the Baire property).

Let Σ be a set, and $A \subseteq \Sigma^\omega$. The *Gale-Stewart game* $\mathbb{G}(\boldsymbol{A})$ [GalSte53] is a two-player infinite game with perfect information where players take turn playing letters from Σ. Player I begins. After ω moves, they produce an infinite word $\alpha \in \Sigma^\omega$. Player I wins iff $\alpha \in A$. A play of this game is illustrated below.

$$\begin{array}{lccccccccc}
\text{I}: & x_0 & & x_2 & \cdots\cdots\cdots & x_n & & x_{n+2} & \cdots\cdots\cdots \\
 & & \searrow \ \nearrow & & \searrow & & \searrow \ \nearrow & & \\
\text{II}: & & x_1 & & \cdots\cdots & & x_{n+1} & & \cdots\cdots
\end{array}$$

Let Σ_A, Σ_B be two sets, and $A \subseteq {\Sigma_A}^\omega$, $B \subseteq {\Sigma_B}^\omega$. The *Wadge game* $\mathbb{W}(\boldsymbol{A}, \boldsymbol{B})$ [Wad72] is a two-player infinite game with perfect information, where player I is in charge of subset A, and player II is in charge of subset B. Players take turn playing letters from Σ_A and Σ_B, respectively. Player I begins. Player II is allowed to skip provided he plays infinitely many letters; player I is not. After ω moves, player I and II have respectively produced two infinite words $\alpha \in {\Sigma_A}^\omega$, and $\beta \in {\Sigma_B}^\omega$. Player II wins in $\mathbb{W}(A, B)$ iff $(\alpha \in A \leftrightarrow \beta \in B)$. A play of this game is illustrated below.

$$\begin{array}{lcccccccc}
(A)\ \text{I}: & a_0 & & a_1 & \cdots\cdots & & \xrightarrow{\text{after } \omega \text{ moves}} & \alpha = a_0 a_1 a_2 \cdots \\
 & & \searrow \ \nearrow & & \searrow & & & \\
(B)\ \text{II}: & & b_0 & & b_1 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & \beta = b_0 b_1 b_2 \cdots
\end{array}$$

3 ω-semigroups

J.-É. Pin introduced the structure of an ω-semigroup [PerPin04] in order to give an algebraic counterpart to the notion of automaton reading infinite words. He showed the equivalence between a finite Büchi automaton and a finite ω-semigroup in the following sense:

– For any finite Büchi automaton $\mathcal{A}$ recognizing the language $L(\mathcal{A})$, one can build a finite ω-semigroup $S_{\mathcal{A}}$ recognizing (in an algebraic sense) the same language $L(\mathcal{A})$.

– For any finite ω-semigroup S recognizing the language $L(S)$, one can build a finite Büchi automaton recognizing the same language $L(S)$.

Definition 3.1. *[PerPin04] An* ω-semigroup *is an algebra consisting in two components,* $\boldsymbol{S = (S_+, S_\omega)}$*, and equipped with the following operations:*

- *A binary operation defined on S_+ and denoted multiplicatively.*
- *A mapping $S_+ \times S_\omega \to S_\omega$ called mixed product, that associates with each pair $(s,t) \in S_+ \times S_\omega$ an element st of S_ω.*
- *A surjective mapping $\pi_S : {S_+}^\omega \to S_\omega$ called infinite product.*

Moreover, these three operations must satisfy the following properties:

1. *S_+ equipped with the binary operation is a semigroup,*
2. *$\forall\, s,t \in S_+ \;\forall\, u \in S_\omega \; s(tu) = (st)u$,*
3. *the infinite product π_S is ω-associative, meaning that for every strictly increasing sequence of integers $(k_n)_{n>0}$, and for every sequence $(s_n)_{n\in\omega}$ $\in {S_+}^\omega$, we have*

$$\pi_S(s_0 s_1 \cdots s_{k_1-1}, s_{k_1} \cdots s_{k_2-1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$

4. *$\forall\, s \in S_+ \;\forall\, (s_n)_{n\in\omega} \in {S_+}^\omega$*

$$s\pi_S(s_0, s_1, s_2, \ldots) = \pi_S(s, s_0, s_1, s_2, \ldots).$$

Intuitively, an ω-semigroup is just a semigroup equipped with a suitable infinite product. It is *finite* precisely when S_+ is finite. Otherwise it is *infinite*. A subset $X \subseteq S_\omega$ is called an ω*-subset*. We focus on those subsets in the sequel.

Definition 3.2. *Let $S = (S_+, S_\omega)$, $T = (T_+, T_\omega)$ be two ω-semigroups. A* morphism of ω-semigroups *from S into T is a pair $\phi = (\phi_+, \phi_\omega)$, where $\phi_+ : S_+ \longrightarrow T_+$ is a morphism of semigroups, and $\phi_\omega : S_\omega \longrightarrow T_\omega$ is a mapping preserving the infinite product, i.e. for every sequence $(s_n)_{n\in\omega}$ of elements of S_+, one has*

$$\phi_\omega\big(\pi_S(s_0, s_1, s_2, \ldots)\big) = \pi_T\big(\phi_+(s_0), \phi_+(s_1), \phi_+(s_2), \ldots\big).$$

Example 3.3. Let A be an alphabet. The ω-semigroup

$$\boldsymbol{A^\infty = (A^+, A^\omega)}$$

equipped with the usual concatenation is the *free ω-semigroup* over alphabet A. It is free in the sense that, for any ω-semigroup $S = (S_+, S_\omega)$, any function f from A into S_+ can uniquely be extended to a morphism of ω-semigroups $\bar{f} = (f_+, f_\omega)$ from A^∞ into S [CarPer97]. We do this by setting $f_+ : A^+ \longrightarrow S_+$ defined by

$$f_+(a_0a_1 \cdots a_n) = f(a_0)f(a_1) \cdots f(a_n) \text{ , with } a_i \in A \ (\forall i \leq n),$$

and $f_\omega : A^\omega \longrightarrow S_\omega$ defined by

$$f_\omega(a_oa_1a_2 \cdots) = \pi_S(f(a_0), f(a_1), f(a_2), \ldots) \text{ , with } a_i \in A \ (\forall i).$$

So, sets of ω-words, in other words sets of reals, are the less constraint ones with regard to the algebraic structure.

In order to state further results, we put the following topology on ω-subsets:

Definition 3.4. *Let $S = (S_+, S_\omega)$ be any ω-semigroup, and $X \subseteq S_\omega$, we set:*

$$X \textit{ is a } \text{basic open} \textit{ if and only if } \pi_S^{-1}(X) \textit{ is an open of } {S_+}^\omega$$

where ${S_+}^\omega$ is equipped with the product topology of the discrete topology on S_+.

Remark 3.5. For any ω-semigroup $S = (S_+, S_\omega)$, the infinite product π_S is a continuous function by definition of the previous topology.

Remark 3.6. At first glance, the topology defined by taking $sS_\omega =_{def} \{st : t \in S_\omega\}$ as a basic open set (for any $s \in S_+$) would look much nicer. Unfortunately, this topology is much too weak for our purpose. Indeed, with this particular topology, in case S_+ is a group, Borel subsets of S_ω come down to the empty set and the whole space; the reason being that, given sS_ω any basic open set, then $S_\omega = ss^{-1}S_\omega \subseteq sS_\omega$, meaning that $sS_\omega = S_\omega$. We certainly need much more than that as we'll see in the last section.

4 An infinite game over ω-semigroups

In this section, we define a reduction relation between ω-subsets by use of an infinite two-player game over ω-semigroups. We then state some general properties of this reduction relation in order to characterize the set hierarchy that it generates.

4.1 Definitions

Definition 4.1. *Let $S = (S_+, S_\omega)$, $T = (T_+, T_\omega)$ be two ω-semigroups, and X, Y be two ω-subsets of S_ω and T_ω, respectively. The infinite two-player game $\mathbb{SG}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as follows: player I is in charge of subset X, player II is in charge of subset Y. Players I and II alternately play elements of S_+ and $T_+ \cup \{\epsilon\}$, respectively. Player I begins, player II is allowed to skip its turn (by playing ϵ) provided he plays infinitely many moves, otherwise he loses the play. Player I cannot skip its turn. After ω moves, players I and II have respectively produced two infinite sequences $\langle s_0, s_1, \ldots\rangle$, and $\langle t_0, t_1, \ldots\rangle$. A play of this game is illustrated below.*

$$\begin{array}{lccccccl} (X)\ I\ : & s_0 & & s_1 & & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & \langle s_0, s_1, s_2, \ldots\rangle \\ & & \searrow \quad \nearrow & & \searrow & & & \\ (Y)\ II\ : & & t_0 & & t_1 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & \langle t_0, t_1, t_2, \ldots\rangle \end{array}$$

The winning condition is the following: player II wins in $\mathbb{SG}(X, Y)$ if and only if

$$\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y$$

where π_S and π_T are the infinite products of S and T respectively, and $\pi_T(t_0, \ldots, t_{n-1}, \epsilon, t_n, \ldots) =_{def} \pi_T(t_0, \ldots, t_{n-1}, t_n, \ldots)$, meaning that the skipping moves of II are not considered in the infinite product.

A strategy *for player II is a mapping $\boldsymbol{\sigma} : {S_+}^+ \to T_+ \cup \{\epsilon\}$. A strategy for player I is defined similarly. A* winning strategy *for a player (w.s.) is a strategy such that the player always wins when using it. We can now define the following reduction relation:*

$$\boldsymbol{X \leq_{SG} Y} \Leftrightarrow_{def} \text{ II has a w.s. in } \mathbb{SG}(X, Y)$$

and of course

$$\boldsymbol{X <_{SG} Y} \Leftrightarrow_{def} X \leq_{SG} Y \text{ but } Y \not\leq_{SG} X$$

$$\boldsymbol{X \equiv_{SG} Y} \Leftrightarrow_{def} X \leq_{SG} Y \text{ and } Y \leq_{SG} X$$

Following the terminology of Wadge games, we set that:

- *an ω-subset X is* self-dual (s.d.) *iff*

$$X \equiv_{SG} X^c$$

where X^c stands for the complement of X. Otherwise, we say that X is non-self-dual (n.s.d.)*;*
- *an ω-subset X is* initializable *iff there exists Y such that*

$$X \equiv_{SG} Y \text{ and } Y \equiv_{SG} s^{-1}Y \ , \ \forall\, s \in S_+$$

where $s^{-1}Y = \{x \in S_\omega : x = \pi_S(u_1, u_2, \ldots) \wedge \pi_S(s, u_1, u_2, \ldots) \in Y\}$. From a playful point of view, a player in charge of a initializable set X in the SG-game never loses his playful strength during the play. Indeed, for any position $s \in S_+$ that he reaches, he remains as strong as at the beginning, when being in charge of the whole subset X.

Example 4.2. Let $S = (S_+, S_\omega)$ be any ω-semigroup, and $X \subseteq S_\omega$, with $X \neq \emptyset, S_\omega$.

- The relation $\emptyset \leq_{SG} X$ holds. Indeed, we give a w.s. for player II in the game $\mathbb{SG}(\emptyset, X)$. At the end of the play, the infinite product of any infinite sequence played by I obviously doesn't belong to $\emptyset$. So the w.s. for II simply consists in playing in order to be outside X at the end of the play (possible, as $X \neq S_\omega$).
- Similarly, the relation $S_\omega \leq_{SG} X$ holds. The w.s. for II in the game $\mathbb{SG}(X, S_\omega)$ consists in in playing in order to be inside X at the end of the play (possible, as $X \neq \emptyset$).
- The relation $\emptyset \not\leq_{SG} S_\omega$ holds. Indeed, at the end of the play, the infinite product of any infinite sequence played by I doesn't belong to $\emptyset$, and the infinite product of any infinite sequence played by II belongs to S_ω, so that II cannot win against I in any case.
- Similarly, the relation $S_\omega \not\leq_{SG} \emptyset$ holds, as there is no possible w.s. for II in the game $\mathbb{SG}(S_\omega, \emptyset)$.

This shows that the empty set and the whole space are non-self-dual sets, since no one is equivalent to its complement. Moreover, any other set reduces to both of them.

4.2 Properties of the SG-relation

Not using yet any determinacy principle for this game, one cannot say much of the SG-relation, except that it is a partial ordering with no particular interesting properties. However, Martin's Borel Determinacy result [Mar75] easily induces Borel Determinacy for SG-games. As it is the case with the Wadge ordering, this property turns the SG-relation into a much more interesting one.

Theorem 4.3. *(Martin) Let Σ be a set. If A is a Borel subset of Σ^ω, then $\mathbb{G}(A)$ is determined.*

Corollary 4.4. *(SG-Borel Determinacy) Let $S = (S_+, S_\omega)$, $T = (T_+, T_\omega)$ be two ω-semigroups, and $X \subseteq S_\omega$, $Y \subseteq T_\omega$ be two Borel ω-subsets. Then $\mathbb{SG}(X, Y)$ is determined.*

Proof. We define a Borel subset $Z \subseteq ({S_+}^\omega \cup {T_+}^\omega \cup \{\epsilon\})^\omega$ such that a player P has a w.s. in $\mathbb{G}(Z)$ iff the same player P has a w.s. in $\mathbb{SG}(X, Y)$. Let p_1 and p_2 be the following continuous projections from $(S_+ \cup T_+ \cup \{\epsilon\})^\omega$ into $(S_+ \cup T_+ \cup \{\epsilon\})^\omega$ defined by $p_1(u_0u_1u_2u_3\ldots) = u_0u_2u_4\ldots$, and $p_2(u_0u_1u_2u_3\ldots) = u_1u_3u_5\ldots$. Let $X', X'', Y', Y'' \subseteq (S_+ \cup T_+ \cup \{\epsilon\})^\omega$ be defined by

$$\begin{aligned}
X' &= \{\alpha = u_0u_1u_2\ldots : \pi_S(u_0, u_2, u_4, \ldots) \in X\} = p_1^{-1}(\pi_S^{-1}(X))\\
X'' &= \{\alpha = u_0u_1u_2\ldots : \pi_S(u_0, u_2, u_4, \ldots) \in X^c\} = p_1^{-1}(\pi_S^{-1}(X^c))\\
Y' &= \{\alpha = u_0u_1u_2\ldots : \pi_T(u_1, u_3, u_5, \ldots) \in Y\} = p_2^{-1}(\pi_T^{-1}(Y))\\
Y'' &= \{\alpha = u_0u_1u_2\ldots : \pi_T(u_1, u_3, u_5, \ldots) \in Y^c\} = p_2^{-1}(\pi_T^{-1}(Y^c))
\end{aligned}$$

By continuity of the functions p_1, p_2, π_S, π_T, these sets are all Borel, and we conclude by taking $Z = (X' \cap Y') \cup (X'' \cap Y'')$.

□

Similarly to the Wadge ordering, and as a consequence of Borel determinacy for these games, come the following interesting results. The first one is an immediate consequence of determinacy. The second one is a corollary of the first one: it states that, for this partial ordering $\leq_{SG}$, the antichains have length at most two. The third one is a result from Martin and Monk establishing the wellfoundness of this $\leq_{SG}$-relation on Borel ω-subsets.

Corollary 4.5. *Let $S = (S_+, S_\omega)$, $T = (T_+, T_\omega)$ be two ω-semigroups, and $X \subseteq S_\omega$, $Y \subseteq T_\omega$ be two Borel ω-subsets. Then*

$$X \not\leq_{SG} Y \Rightarrow Y \leq_{SG} X^c.$$

Proof. The relation $X \not\leq_{SG} Y$ means that player II doesn't have a winning strategy in $\mathbb{SG}(X, Y)$. Hence, by determinacy, player I has a winning strategy σ in this game. So Player II has the following winning strategy in $\mathbb{SG}(Y, X^c)$: he copies the first move of player I in $\mathbb{SG}(X, Y)$, and then, at each step n, he plays $\sigma(x_0 \cdots x_n)$, where $x_0, \ldots, x_n$ are the moves already played by I in $\mathbb{SG}(Y, X^c)$.

□

Corollary 4.6. *(Wadge's lemma) Let $S = (S_+, S_\omega)$, $T = (T_+, T_\omega)$ be two ω-semigroups, and $X \subseteq S_\omega$, $Y \subseteq T_\omega$ be two Borel ω-subsets. Then only one of these possibilities occurs:*

- *$X \leq_{SG} Y$ and $Y \not\leq_{SG} X$, which implies $X <_{SG} Y$.*
- *$X \leq_{SG} Y$ and $Y \leq_{SG} X$, which implies $X \equiv_{SG} Y$.*
- *$X \not\leq_{SG} Y$ and $Y \not\leq_{SG} X$, which implies $X \equiv_{SG} Y^c$.*
- *$X \not\leq_{SG} Y$ and $Y \leq_{SG} X$, which implies $Y <_{SG} X$.*

Proof. The first, second and fourth cases come from the very definition. The third case comes by the previous proposition, and by the obvious fact that $A \leq_{SG} B \Leftrightarrow A^c \leq_{SG} B^c$ holds, for any ω-subset A and B.

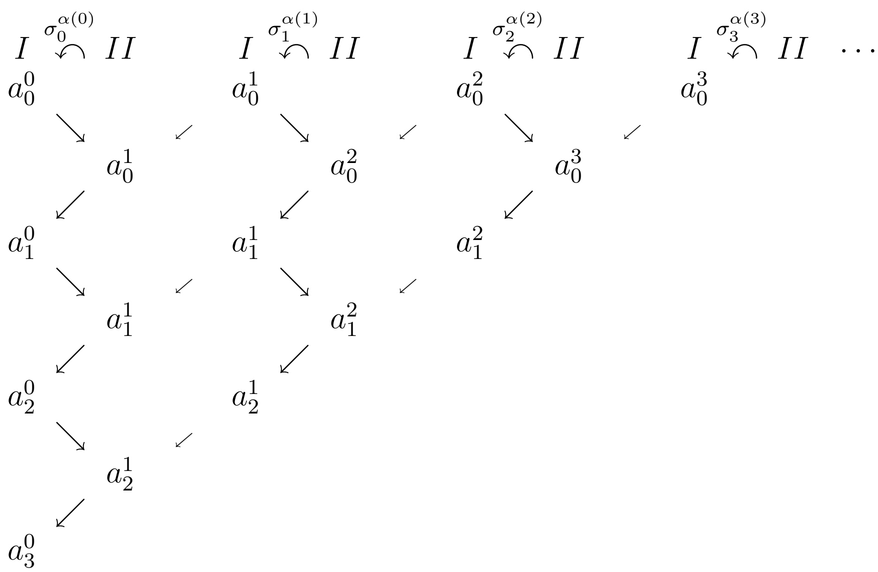
□

Proposition 4.7. *(Martin, Monk) The partial ordering $<_{SG}$ is wellfounded on Borel ω-subsets, meaning that there is no infinite sequence of Borel ω-subsets $(A_i)_{i\in\omega}$ such that*

$$A_0 >_{SG} A_1 >_{SG} \ldots >_{SG} A_n >_{SG} A_{n+1} >_{SG} \ldots$$

Proof. Towards contradiction, assume that there exists an infinite sequence of ω-semigroups $\{S_i = (S_{i,+}, S_{i,\omega})\}_{i\in\omega}$, and an infinite strictly $<_{SG}$-descending sequence of Borel ω-subsets $(A_n)_{n\in\omega}$, where $A_i \subseteq S_{i,\omega}$, (any $i \in \omega$). For all $n \geq 0$, the relation $A_n >_{SG} A_{n+1}$ implies that both $A_n \not\leq_{SG} A_{n+1}$ and $A_n^c \not\leq_{SG} A_{n+1}$ hold, meaning that player I has w.s. σ_n^0 and σ_n^1 in both games $\mathbb{SG}(A_n, A_{n+1})$ and $\mathbb{SG}(A_n^c, A_{n+1})$, respectively. Let $\alpha \in 2^\omega$ define the following sequence of strategies $(\sigma_k^{\alpha(k)})_{k\in\omega}$.

We now consider ω many SG-games linked this way: in the first game, player I applies strategy $\sigma_0^{\alpha(0)}$ to II's play. Since it is a strategy for I, it gives the first letter a_0^0 before II has ever played anything, but then, applying $\sigma_0^{\alpha(0)}$ means to know II's first move a_0^1. Precisely, II copies I's moves in the second game, in which I applies the w.s. $\sigma_1^{\alpha(1)}$. And so on for every game. This means, in game number n, player I applies strategy $\sigma_n^{\alpha(n)}$, and II scrupulously copies I's moves in the game number $n+1$. These ω many games chained together are illustrated below. Big arrows denote the action of playing while little ones denote the action of copying.



Let $x_\alpha = \Pi_{k\in\omega} a_k^0$ be the infinite word played by player I in the first game, $\phi : 2^\omega \to {S_{0,+}}^\omega$ defined by $\phi(\alpha) = x_\alpha$, and $\psi = \pi_{S_0} \circ \phi : 2^\omega \to S_{0,\omega}$ defined by $\psi(\alpha) = \pi_{S_0}(x_\alpha) = \pi_{S_0}(\Pi_{k\in\omega} a_k^0)$. By definition of these chained games, ϕ is continuous. Indeed, we remark that the k first letters of x_α only depend on the k first letters of α, as we completely don't need games number $k+1, k+2, \ldots$ to determine $x_\alpha \restriction k$. So, for any $U \subseteq {S_{0,+}}^*$, $\phi^{-1}(U{S_{0,+}}^\omega) = V2^\omega$, with $V \subseteq 2^*$, meaning that the pre-image by ϕ of a basic open set is a basic open set. As ϕ and π_{S_0} are continuous, so is ψ. Consider $B = \psi^{-1}(A_0)$. By construction of these chained games, we notice that if α and α' only differ by one position (i.e. $\exists!\ i\ s.t.\ \alpha(i) \neq \alpha'(i)$), then $\alpha \in B \Leftrightarrow \alpha' \notin B$. This means that B is a flip set, and it is Borel as ψ is continuous, a contradiction.

□

Remark 4.8. Quotienting Borel ω-subsets by the equivalence relation $\equiv_{SG}$, leads to a hierarchy of classes of Borel ω-subsets called the SG-*hierarchy*. As already mentioned, the previous results state the wellfoundness of this hierarchy together with the fact that the antichains have length at most two. The SG-hierarchy has thus the same familiar ”scaling shape” as the Borel hierarchy or the Wadge hierarchy: an increasing sequence of pairs of non-self-dual classes with single self-dual classes in between. This hierarchy is illustrated in figure 1. Circles represent classes of Borel ω-subsets, and arrows represent the fact of ”being SG-smaller than”.

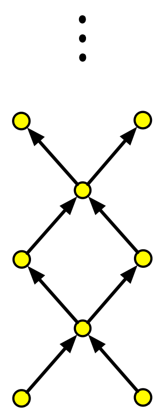


Fig. 1. the SG-hierarchy

Definition 4.9. *The* SG-degree *of Borel ω-subsets is defined by induction. At the bottom, we find $\emptyset$ and $\emptyset^c$ since there is no non-empty set A such that $A \leq_{SG} \emptyset$ holds, and there is also no other smaller set than the whole space, which is incomparable to the empty set (see example 4.2). So we set:*

$$d^o_{SG}(\emptyset) = d^o_{SG}(\emptyset^c) = 0,$$

and for any Borel ω-subset $A >_{SG} \emptyset$

$$d^o_{SG}(A) = \sup\{d^o_{SG}(B) + 1 : B <_{SG} A\}.$$

5 Basic results about this game

In this section, we give some general results about both this infinite game over ω-semigroups, and more precisely about the SG-hierarchy. We state that two important hierarchies become particular cases of the SG-hierarchy. But the most striking thing is that very essential algebraic notions turn out to correspond to very natural properties stated in a game theoretical way.

5.1 The Wadge hierarchy

In the late sixties, W. W. Wadge introduced a very deep refinement of the Borel hierarchy of sets of the Baire space (or of the Cantor space as well) [Wad72]. The *Wadge hierarchy* is induced by the following relation on sets: $A \leq_W B \Leftrightarrow_{def} \exists f$ continuous s.t. $f^{-1}(B) = A \Leftrightarrow$ II has a w.s. in $\mathbb{W}(A, B)$ [Wad72].

Proposition 5.1. *The SG-hierarchy restricted to Borel ω-subsets of free ω-semigroups corresponds exactly to the Wadge hierarchy of Borel subsets.*

Proof. When restricted to free ω-semigroups, the SG-game is exactly the same as the Wadge game.

□

Remark 5.2. As a matter of fact, the SG-hierarchy should be regarded as a widening of the Wadge hierarchy. Not only more sets are involved, but the algebraic structure of semigroups enriches the way one can describe or characterize Borel sets. For instance, some of them may ”live” in an ω-semigroup generated by a monoid, or even group, while most don’t.

5.2 The Wagner hierarchy

In 1979, Klaus Wagner described a hierarchy among languages recognized by Muller automata called the *Wagner hierarchy* [Wag79]. This hierarchy has height ω^ω and actually coincides with the restriction of the Wadge hierarchy to ω-rational languages. In other words, it is the hierarchy induced by the following ordering on Muller automata: $\mathcal{A} \leq_W \mathcal{B}$ iff the language recognized by $\mathcal{A}$ is the inverse image of the language recognized by $\mathcal{B}$ by a continuous function. This section shows that the Wagner hierarchy is a particular case of the SG-hierarchy.

Proposition 5.3. *The SG-hierarchy restricted to subsets of finite ω-semigroups is classwise isomorphic to the Wagner hierarchy.*

Proof. In the forthcoming paper [CabDup0?].

□

The decidability of the Wagner hierarchy also holds in the following sense:

Proposition 5.4. *Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and $X \subseteq S_\omega$ be Borel. One can associate to X an ordinal $\xi_X \in \omega^\omega$ being its degree in the Wagner hierarchy.*

Proof. In the forthcoming paper [CabDup0?].

□

5.3 Basic algebraic properties

Important algebraic notions can be expressed in a natural game theoretical way by use of the SG-game. These results militate in favor of developing the use of game theoretical tools in algebra. The two following propositions give a game theoretical approach of the algebraic concepts of monoid and group.

Proposition 5.5. *Let $S = (S_+, S_\omega)$ be any ω-semigroup, and $X \subseteq S_\omega$ be any Borel ω-subset. The following conditions are equivalent:*

(1) *$X \not\leq_{SG} X^c$ (i.e. X is n.s.d.).*
(2) *Every player in charge of X in the SG-game is allowed to skip his turn, provided he plays infinitely many letters, otherwise he loses.*
(3) *There exists an ω-semigroup $T = (T_+, T_\omega)$ and a Borel ω-subset $Y \subseteq T_\omega$ such that T_+ is a monoid, and $X \equiv_{SG} Y$.*

Proof. (sketch)

$(1) \Rightarrow (2)$: We show that we can assume without loss of generality that any player in charge of X in the SG-game can skip his turn, provided he plays infinitely many letters. In other words, we show that a player in charge of X that is allowed to skip is not stronger than (or can be beaten by) a player in charge of X that is not allowed to. Let $\overline{SG}(\_,\_)$ be the same infinite game as $SG(\_,\_)$, instead that player I is allowed to skip - provided he plays infinitely often - while player II is not. By hypothesis, there exists a winning strategy σ for I in the game $SG(X, X^c)$. Then σ is also a winning strategy for II in the game $\overline{SG}(X, X)$.

$(2) \Rightarrow (1)$: By hypothesis, every player in charge of X is allowed to skip its turn, provided he plays infinitely letters. The winning strategy for player I in the game $\mathbb{SG}(X, X^c)$ consists in skipping the first move, and then copy player II.

$(3) \Rightarrow (1)$: By hypothesis, $X \equiv_{SG} Y$, with $Y \subseteq T_+$, and T_+ is a monoid. We thus show that Y is non-self-dual by giving a winning strategy for I in the game $SG(Y, Y^c)$: player I fist plays $\mathbf{1}$; then when II doesn't skip, I copies II, and when II skips, I plays $\mathbf{1}$. As Y is non-self-dual, so is X.

$(1) \Rightarrow (3)$: A consequence of [Dup01] and [Dup0?]. Basically, the idea is to consider the set $Z = \pi_S^{-1}(X)$. Viewed as a subset of the free ω.semigroup $({S_+}^+, {S_+}^\omega)$ - with ${S_+}^\omega$ equipped with the usual topology (the product topology of the discrete topology over S_+) - it satisfies $Z \equiv_{SG} X$. Since Z is Borel and non self dual, it follows from both [Dup01], and [Dup0?] that there exists some $\bar{Y} \subseteq {S_+}^{\leq\omega}$ verifying:

- $\bar{Y}^b \equiv_W Z$, where $\bar{Y}^b$ stands for all ω-sequences x built over the alphabet $S_+ \cup \{b\}$ - where b stands for any new letter not in S_+ - that verify: "x in which every occurrence of the letter b has been erased, belongs to $\bar{Y}$."
- $\bar{Y} \cap {S_+}^\omega = Z$

As $\bar{Y}^b \equiv_W Z$ holds, then $\bar{Y}^b \equiv_{SG} Z$, when these sets are considered as subsets of the free ω-semigroups $\big((S_+ \cup \{b\})^+, (S_+ \cup \{b\})^\omega\big)$, and $({S_+}^+, {S_+}^\omega)$, respectively. As $Z \equiv_{SG} X$ also holds, then $\bar{Y}^b \equiv_{SG} X$. By identifying b and the identity element, i.e. by setting the monoid $T_+ = (S_+ \cup \{b\})^+ = (S_+ \cup \{\mathbf{1}\})^+, T_\omega = (S_+ \cup \{b\})^\omega = (S_+ \cup \{\mathbf{1}\})^\omega$, and by taking $Y = \bar{Y}^b \subseteq T_\omega$, one gets the result.

□

Proposition 5.6. *Let $S = (S_+, S_\omega)$ be any ω semigroup, and $X \subseteq S_\omega$ be any Borel subset. The following conditions are equivalent:*

(1) $X \leq_{SG} s^{-1}X, \forall\, s \in S_+$ *(i.e. X is initializable).*

(2) *Every player in charge of X in the SG-game is allowed to erase his moves, provided he plays infinitely many letters, otherwise he loses.*
(3) *There exists an ω-semigroup $T = (T_+, T_\omega)$ and a Borel ω-subset $Y \subseteq T_\omega$ such that T_+ is a group, and $X \equiv_{SG} Y$.*

Proof. (sketch)

$(3) \Rightarrow (2):$ We show that we can assume without loss of generality that any player in charge of X in the SG-game can erase his moves, provided he plays infinitely many letters. In other words, we show that a player in charge of X that is allowed to erase is not stronger than (or can be beaten by) a player in charge of X that is not allowed to. Let $\tilde{SG}(\_,\_)$ be the same infinite game as $SG(\_,\_)$, instead that player I is allowed to erase his moves - provided he plays infinitely often - while player II is not. We first show that player II has a w.s. in $\tilde{SG}(X, Y)$. By hypothesis, II has a w.s. σ in the game $SG(X, Y)$. This leads the following w.s. for II in $\tilde{SG}(X, Y)$: II copies I, and when I erases a part of his position, then II "cancels" a piece of his by playing the suitable inverse element, in order to come back to the expected situation. By hypothesis, II also has a winning strategy in the game $SG(Y, X)$ (where no one can erase his moves). Then by composition of strategies, II has a winning strategy in the game $\tilde{SG}(X, X)$.

$(2) \Rightarrow (1):$ Let $s \in S_+$. By hypothesis, we can give the following winning strategy σ in the game $SG(X, X)$, but where player II has already played the element s: player II erases s, and then copies player I. By the prevous point, we can find a winning srategy σ' in the game $\tilde{SG}(X, X)$ (where I can erase, while II cannot). The composition of these strategies $\sigma'' = \sigma' \circ \sigma$ is winning in the game $SG(X, s^{-1}X)$.

$(1) \Rightarrow (3):$ A consequence of [Dup01] and [Dup0?]. First, since X is clearly non-self-dual, one can assume w.l.o.g. that S_+ is a monoid with $\mathbf{1}$ as identity (otherwise, from previous proposition, one can get some X' satisfying this property). Then, here also, the idea is to consider the set $Z = \pi_S^{-1}(X)$. Viewed as a subset of the free ω-semigroup $({S_+}^+, {S_+}^\omega)$ - with ${S_+}^\omega$ equipped with the usual topology - one has $X \equiv_{SG} Z$. Since Z is Borel and initializable, from [Dup01] and [Dup0?], we know that there exists some set $B \subseteq \{0, 1\}^{\leq\omega}$ such that:

- $(B^\sim)^b \equiv_W Z$, where $B^\sim$ is defined as B plus an additional eraser, and B^b is defined as $\bar{Y}^b$ was in last proposition (b stands for "blank", it behaves just like a mute letter). In a few words, this means that a player in charge of $(B^{\sim})^b$ in a Wadge game (either player I or player II) is like a player in charge of B with two extra possibilities. This player can:

- play b, which is just like skipping, except that here, one can decide to skip forever, which is materialized by playing infinitely many b's;
- erase all or part of his/her last moves (b is just like a skip, it doesn't count as a true letter).

After ω such moves, the resulting sequence played is the limit of what has been played, forgetting about the blanks. And $(B^{\sim})^b$ is the set of all infinite sequences that can possibly be played such that their limits belong to B.

– Now, add two more letters 0^{-1}, and 1^{-1} viewed as the inverse elements of respectively 0, and 1. Consider the free semigroup $\{0, 1, 0^{-1}, 1^{-1}, b\}^*$, where the concatenation operation moreover verifies $0^{-1}0 = 00^{-1} = b$, and $1^{-1}1 = 11^{-1} = b$. Take $b = \mathbf{1}$, and set Y as the set of all infinite sequences over $\{0, 1, 0^{-1}, 1^{-1}, \mathbf{1}\}$, such that, once every possible "erasing" of the form $0^{-1}0 = 00^{-1} = \mathbf{1}$, or $1^{-1}1 = 11^{-1} = \mathbf{1}$ has been processed, yields an infinite sequence that belongs to B^b (which is $B^{\mathbf{1}}$, since $b = \mathbf{1}$), if one forgets about the subscripts $^{-1}$ (i.e. identifying 0^{-1} with 0 and 1^{-1} with 1). It is easy to see that $(B^{\sim})^b \equiv_W Y$.

One gets the result by considering the ω-semigroup $T = (T_+, T_\omega) = \left(\{0, 1, 0^{-1}, 1^{-1}, \mathbf{1}\}^+, \{0, 1, 0^{-1}, 1^{-1}, \mathbf{1}\}^\omega\right)$, and the subset $Y \subseteq T_\omega$ as defined above. Indeed, one has $Z \equiv_W (B^{\sim})^b \equiv_W Y$, meaning that $Z \equiv_{SG} Y$ (by treating Z, and Y as subsets of the suitable ω-semigroups). So, $X \equiv_{SG} Z \equiv_{SG} Y$.

□

6 Conclusion

The way we see it, further developments in the Wadge hierarchy, for instance, should be deeply related to the SG-hierarchy. It seems to be of a major interest to be able to characterize a Borel set of reals, by the type of ω-semigroups where a complete set for the Wadge class it generates may "live". This should be a way of identifying the algebraic properties hidden behind various "Borel attitudes" of sets. In other words, an algebraic way of classifying Borel sets. A very promising approach.

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