The Algebraic Counterpart of the Wagner Hierarchy

Jérémie Cabessa

University of Lausanne

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Classifying ω-rational languages

Automata
The Wagner Hierarchy
[Wag79]

Descriptive set theory
The Wadge hierarchy
of ω-rational languages
[Sel98]

Algebra
[CarPer96,99], [DupRis06]
The FSG-hierarchy

The Wagner Hierarchy
(most refined classification of ω-rational languages)

Given two ω-rational languages K and L, we set

$$K \leq_W L \quad \text{iff} \quad K = f^{-1}(L) \text{ for some continuous function } f$$
$$\text{iff} \quad \text{Player II has a w.s. in the Wadge game } W(K,L)$$

Then as usual, $K <_W L$ iff $K \leq_W L \not\leq_W K$ and $K \equiv_W L$ iff $K \leq_W L \leq_W K$.

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The collection of all ω-rational languages ordered by the relation $\leq_W$ is called *the Wagner hierarchy*.

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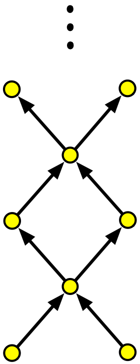

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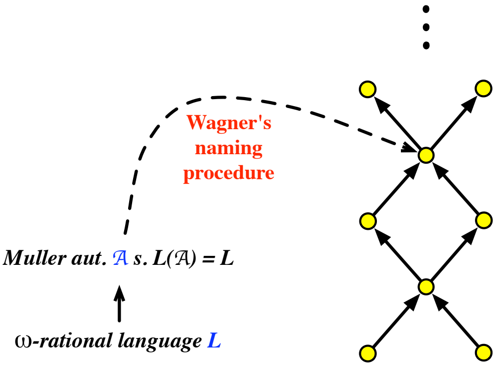
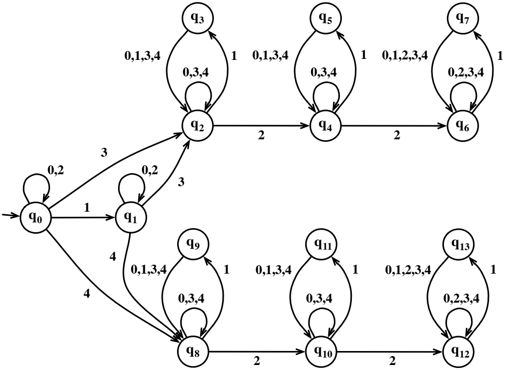


Let L be recognized by the following Muller automata



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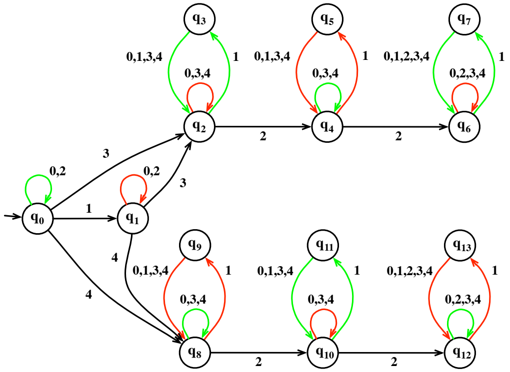

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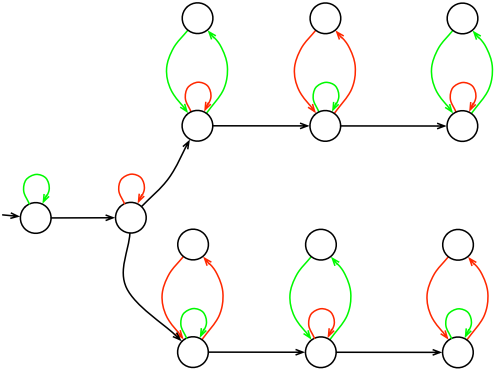

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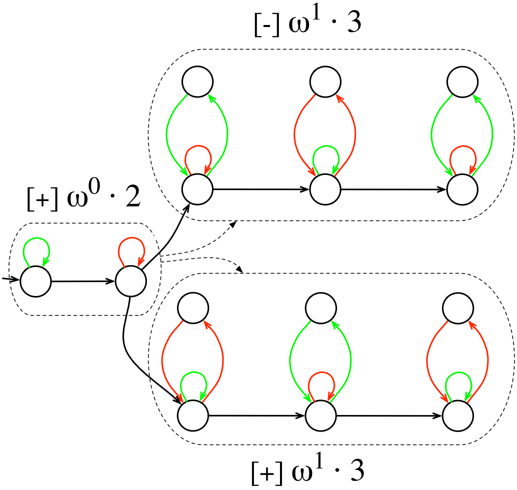

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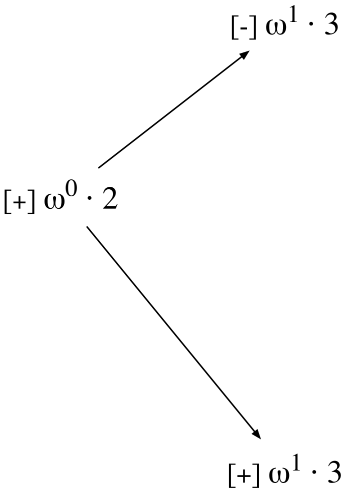

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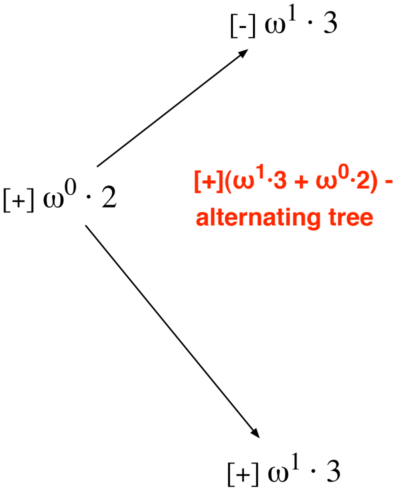

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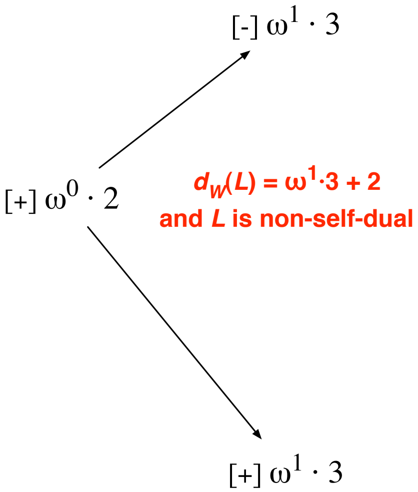

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Theorem (Wagner)

Let L be an ω-rational language and $\alpha < \omega^\omega$. Then $d_W(L) = \alpha$ iff any Muller $\mathcal{A}(L)$ automata recognizing L contains a maximal α-alternating tree.

rational languages

finite automata $\Longleftrightarrow$ **finite semigroups**

$a(bc) = (ab)c$

ω-rational languages

finite ω-automata

finite ω-semigroups

$a(bc) = (ab)c$ & infinite product

Definition (ω-semigroup)

An ω-semigroup S is a pair (S_+, S_ω), where S_+ is a semigroup, S_ω is a set, and equipped with

- an associative mixed product : $S_+ \times S_\omega \longrightarrow S_\omega$
- a surjective infinite product $\pi_S : S_+^\omega \longrightarrow S_\omega$ satisfying

$$\pi_S(s_0 s_1 \cdots s_{k_1 - 1}, s_{k_1} \cdots s_{k_2 - 1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$
$$s\pi_S(s_0, s_1, s_2, \ldots) = \pi_S(s, s_0, s_1, s_2, \ldots).$$

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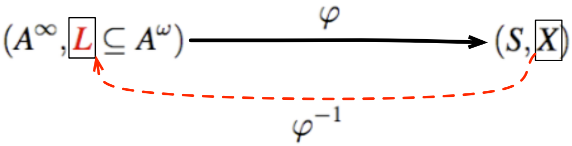

satisfying

$$L = \varphi^{-1}(X)$$

Finite pointed ω-semigroups are the algebraic counterparts of Büchi automata.

Theorem

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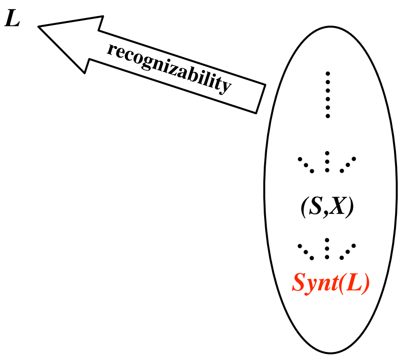
Among all finite pointed ω-semigroups recognizing a given ω-rational language L, there exists a minimal one, **the syntactic ω-semigroup** of L, denoted by $\boldsymbol{Synt(L)}$.



Example

Consider the language $K = ((a+b)^* a)^\omega$. Then $Synt(K) = (S, X)$, where

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0 \cdot 0 = 0 \quad 0 \cdot 1 = 0 \quad 1 \cdot 0 = 0 \quad 1 \cdot 1 = 1$$
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Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two finite ω-semigroups, and let also $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.

The infinite two-player game $\mathbb{SG}((\boldsymbol{S}; \boldsymbol{X}), (\boldsymbol{T}, \boldsymbol{Y}))$ is defined as follows:

- Player I plays elements from S_+,
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- Player II wins the game $\mathbb{SG}(X, Y)$ iff

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We write $(S, X) \leq_{SG} (T, Y)$ iff Player II has a winning strategy in $\mathbb{SG}(X, Y)$. And as usual $(S, X) \equiv_{SG} (T, Y)$ iff $(S, X) \leq_{SG} (T, Y) \leq_{SG} (S, X)$.

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The collection of finite pointed ω-semigroups ordered by the $\leq_{SG}$-relation is called *the* $\mathbb{FSG}$*-hierarchy*.

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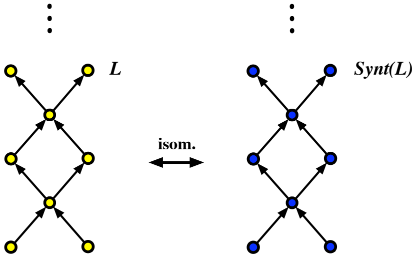

Corollary

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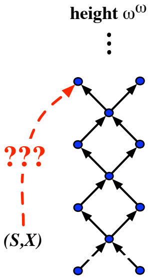

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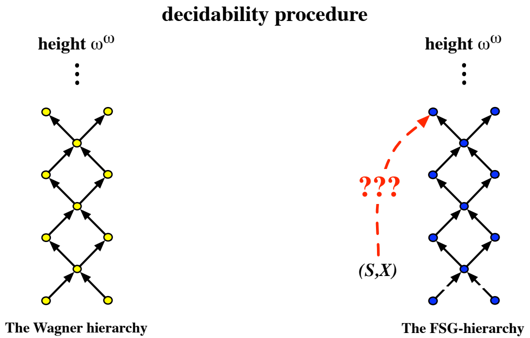

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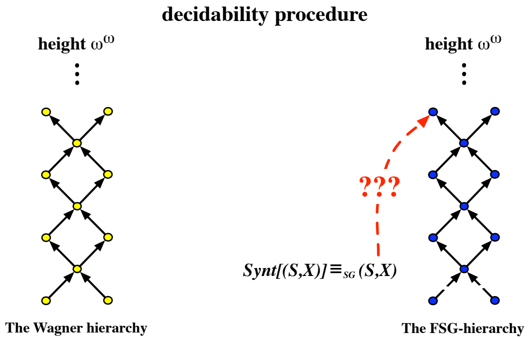

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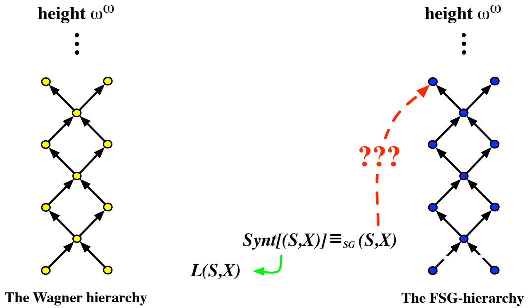

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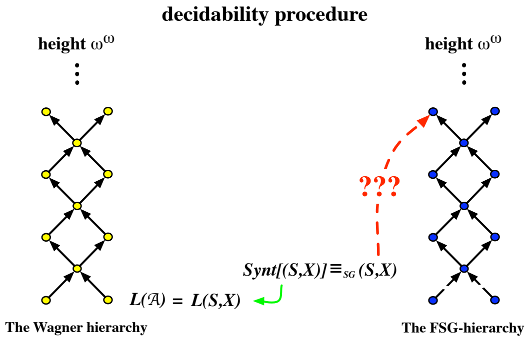

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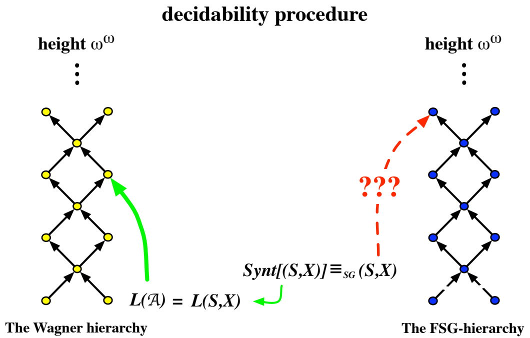

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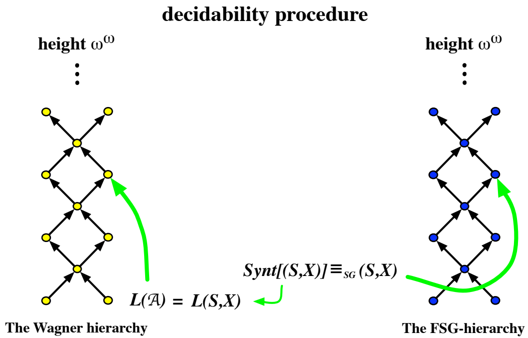

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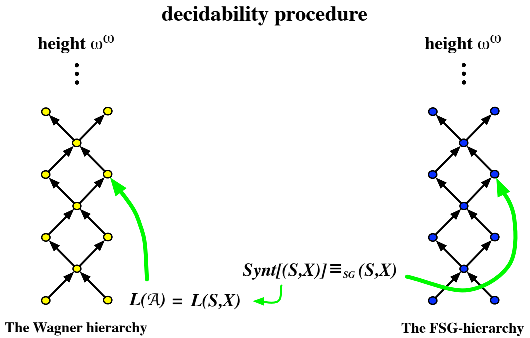

Corollary

Let L be an ω-rational language. Then $d_W(L) = \alpha$ iff $d_{FSG}(Synt(L)) = \alpha$.

Example

Consider the syntactic pointed ω-semigroup of $K = ((a+b)^*a)^\omega$:

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 0 \qquad 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1$$

$$00^\omega = 0^\omega \qquad 10^\omega = 0^\omega \qquad 01^\omega = 1^\omega \qquad 11^\omega = 1^\omega$$

- $X = \{0^\omega\}$.

Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

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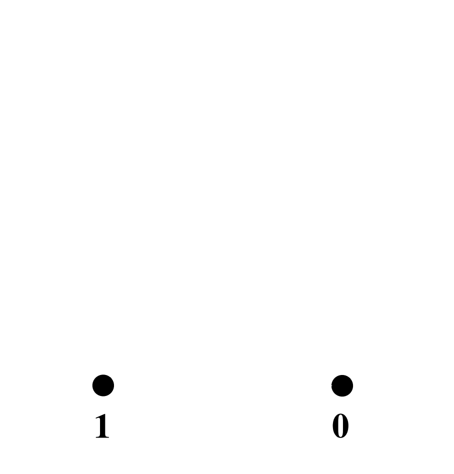

Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example (continued)

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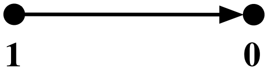

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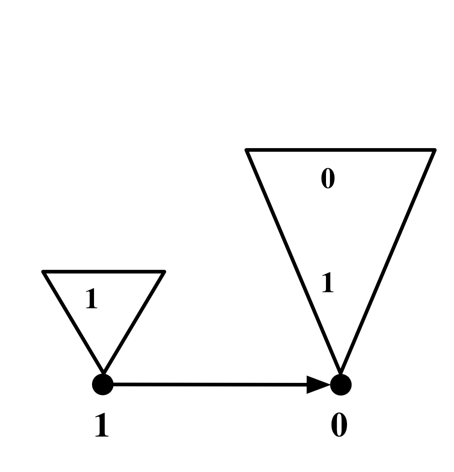

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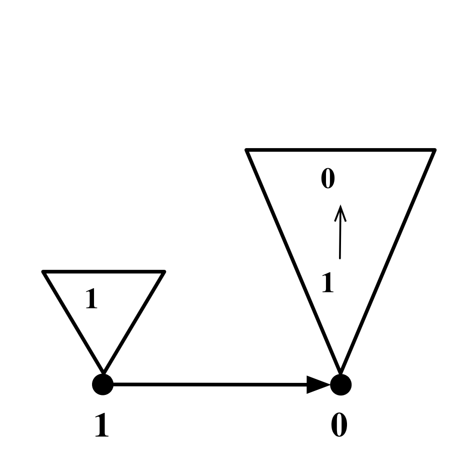

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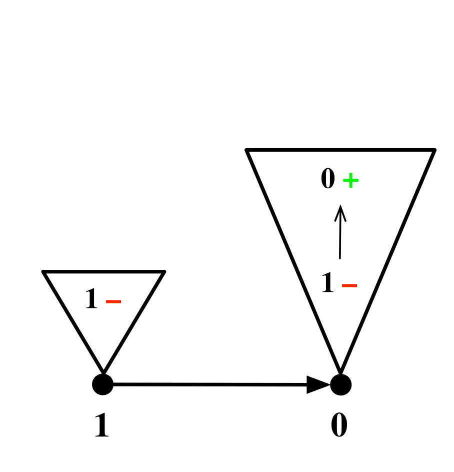

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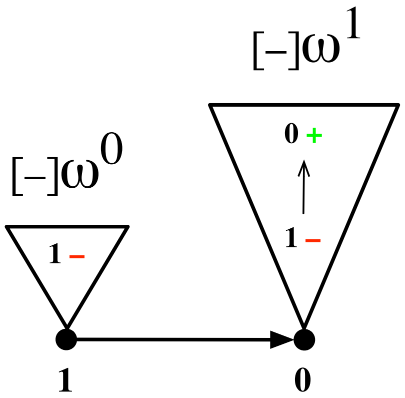

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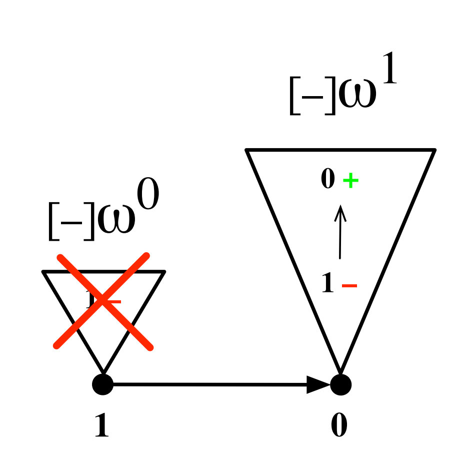

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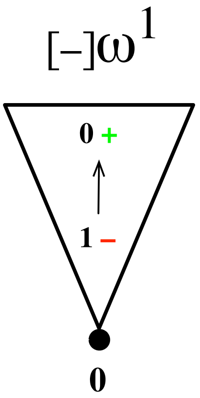

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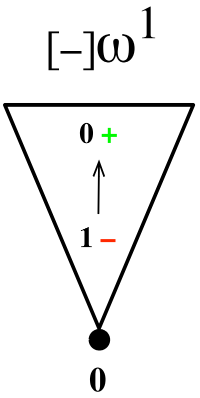

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Example

Consider the syntactic pointed ω-semigroup of $L = (a\{b,c\}^* \cup \{b\})^\omega$:

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

$$
\begin{array}{cccc}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega
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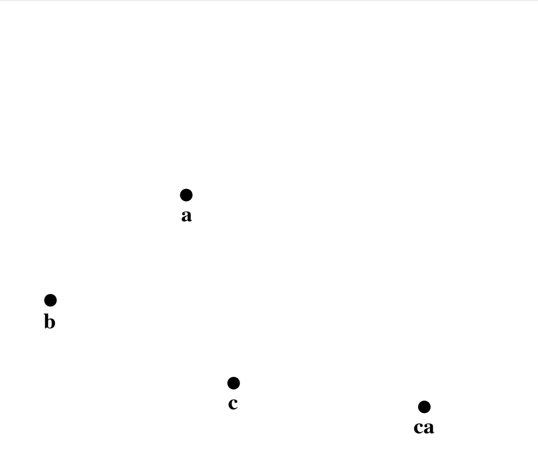

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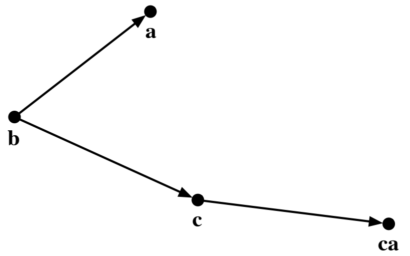

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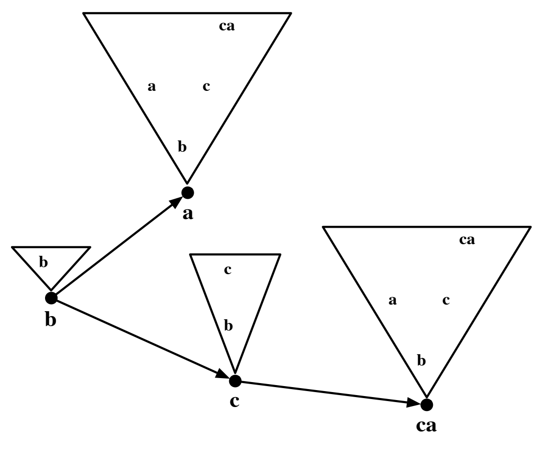

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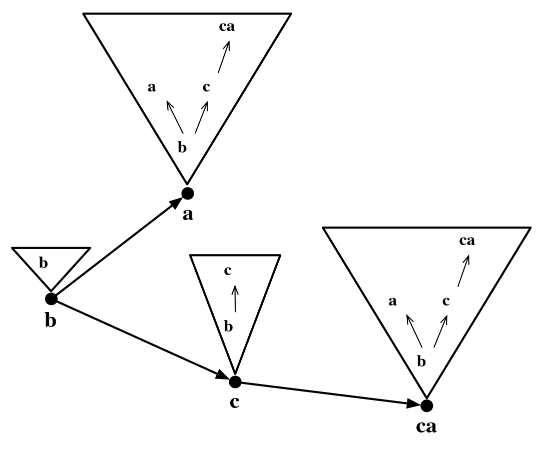

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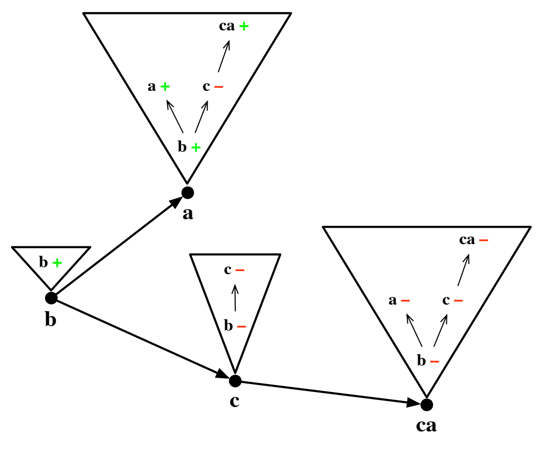

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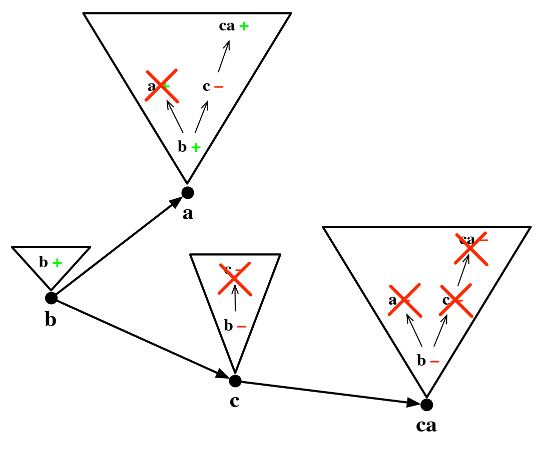

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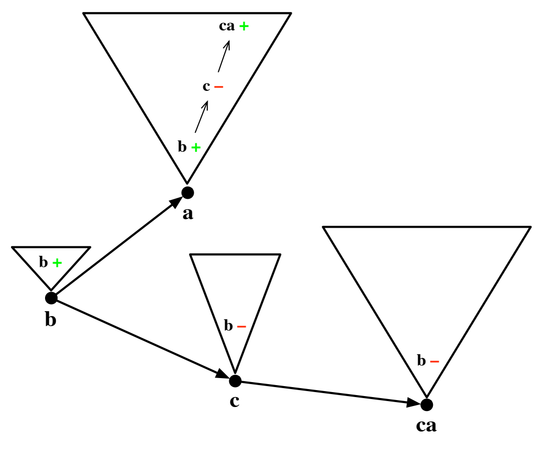

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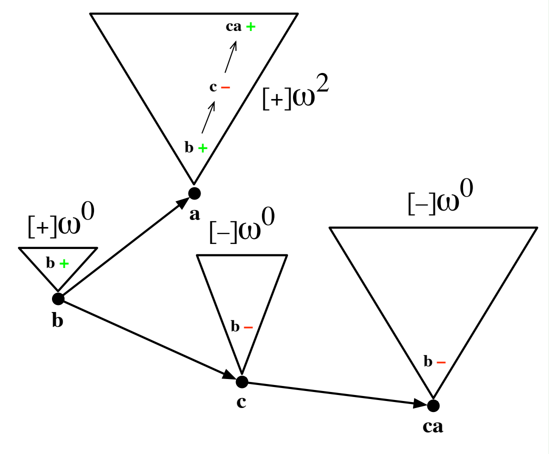

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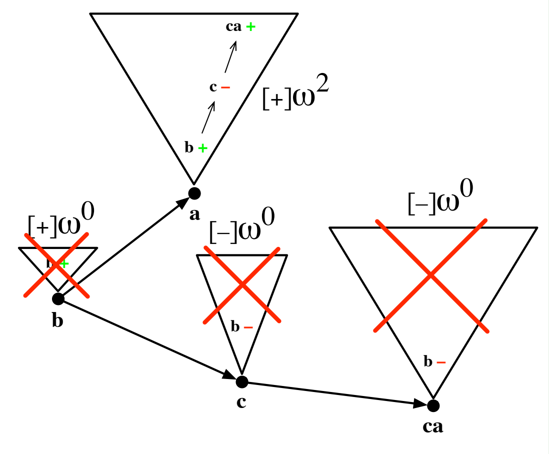

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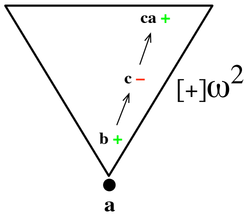

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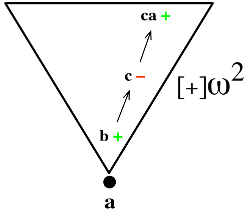

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Proposition

Let (S, X) be a finite pointed ω-semigroup. $d_{FSG}((S, X)) = \alpha$ iff (S, X) contains a maximal α-alternating tree.

Corollary

Let L be an ω-rational language. Then $d_W(L) = \alpha$ iff any finite pointed ω semigroup recognizing L contains a maximal α-alternating tree.

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Summary :
The Wagner hierarchy

From the **automata** point of view

Theorem

Then $d_W(L) = \alpha$ iff any Muller automata recognizing L contains a maximal α-alternating tree.

From the **algebraic** point of view

Theorem

Then $d_W(L) = \alpha$ iff any finite ω-semigroup recognizing L contains a maximal α-alternating tree.

- The $\mathbb{FSG}$-hierarchy is the algebraic counterpart of the Wagner hierarchy.
- Decidability procedure of the $\mathbb{FSG}$-hierarchy.
- **One can compute the Wagner degree of an ω-rational language directly on its syntactic image.**
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