Automata, Semigroups, Games and Logic.

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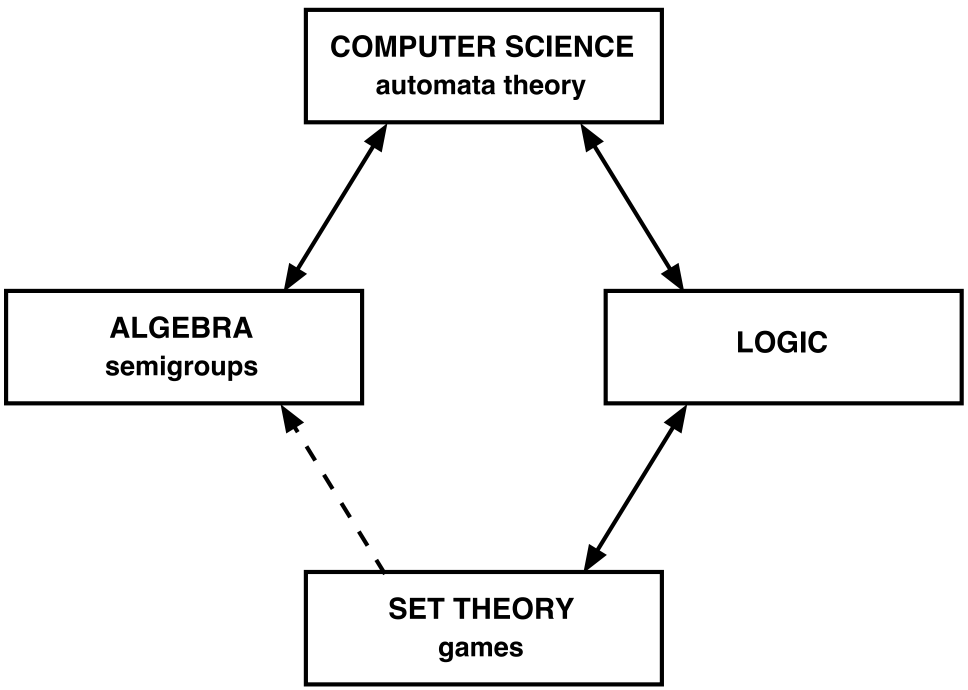
Overview

- Introduction
- Automata and Semigroups
- Automata and Logic
- Games and Logic
- Conclusion

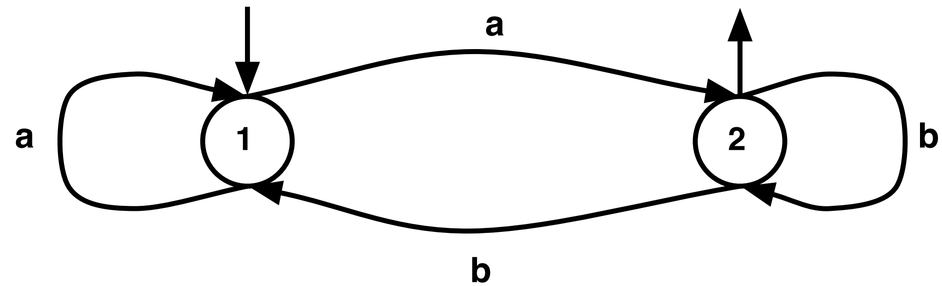
Introduction

- automata $\rightarrow$ many applications: computer science, electronics, linguistics, pure maths ...
- relationships between topics



Automata and Semigroups

- **automaton $\mathcal{A}$ on finite words recognizing language $L^+(\mathcal{A})$**



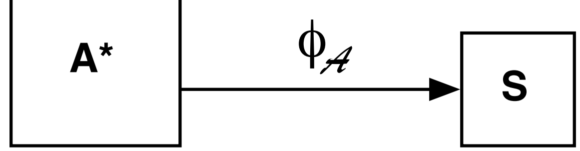
$$L^+(\mathcal{A}) = a\{a,b\}^*$$

- **semigroup $(S,\cdot)$: S a set and $\cdot$ an associative law**

$$(A^+,\cdot) = (\{\text{finite words on the alphabet } A\}, \text{concatenation})$$

automaton $\Rightarrow$ semigroup

From an aut. $\mathcal{A}$ **recog. language** $L^+(\mathcal{A}) \subset A^*$
we associate a semigroup S **and morphism** $\phi_{\mathcal{A}}$



$\phi_{\mathcal{A}}$ **simulates the behaviour of** $\mathcal{A}$**.**

construction: $S = \{\textbf{binary rel. on } Q\}$, **where** $Q = \{\textbf{states of } \mathcal{A}\}$

$$\phi_{\mathcal{A}}(a) = \{(p, q) \in Q \times Q : (p, a, q) \textbf{ is a trans. of } \mathcal{A}\}$$

semigroup $\Rightarrow$ automaton

From a semigroup S and a morphism $\phi : A^* \longrightarrow S$ rec. $L \subset A^*$ we associate automaton $\mathcal{A}_\phi$ s.t.

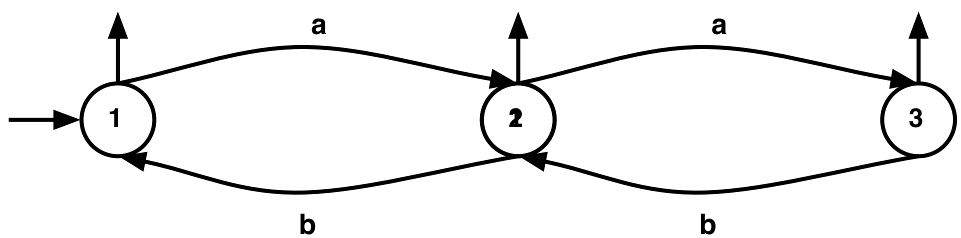
construction: **states** $=$ **elements of** S

trans. $q * a = q \cdot \phi(a)$

Theorem

equivalence between automata recognizing finite words and finite semigroups

- **automaton $\mathcal{A}$ on infinite words recognizing language $L^\omega(\mathcal{A})$**



Büchi condition: $\mathcal{F} = \{1, 2, 3\}$

Muller condition: $\mathcal{F} = \{\{1, 2\}, \{2, 3\}\}$

- ω**-semigroup** $S = (S_+, S_\omega)$
  S_+ **is a finite semigroup**
  S_ω **is a set s.t. every infinite product of elements of** S_+ **belong to** S_ω

  $S = (\{0, 1\}, \{a\})$, **with every infinite product equal to** a.

Theorem

equivalence between automata recognizing infinite words and ω**-semigroups**

Automata and Logic

- $MF_2(<)$ **monadic sec. order logic with predicate symbol "$<$"**

$$\phi = \forall X(\exists x \forall y(X(x) \wedge x < y))$$

$\rightarrow$ **semantic: models are finite or infinite words!**

$$u \textbf{ word of } A^\infty \Rightarrow \mathcal{M}_u = (Dom(u), (\mathbf{a})_{a \in A})$$

$$Dom(u) = \{0, 1, ..., |u|\}\textbf{,}\ \mathbf{a} = \{i < |u| : u(i) = a\}$$

$$u = abbaab \Rightarrow \mathcal{M}_u = (\{0, 1, ..., 6\}, \mathbf{a} = \{0, 3, 4\}), \mathbf{b} = \{1, 2, 5\})$$

automaton $\Rightarrow$ logic $MF_2(<)$

Proposition

any language recognized by a Büchi automaton can be expressed by a formula of $MF_2(<)$

proof: build a formula s.t. the words (i.e models) satisfying it are exactely the words recognized by the automaton.

logic $MF_2(<) \Rightarrow$ automaton

Proposition

any subset of words defined by a formula of $MF_2(<)$ can be recognized by a Büchi automaton

constructive proof: subset of words defined by the formula is *built* inductively, we check at each step that it is recognizable by a Büchi automaton

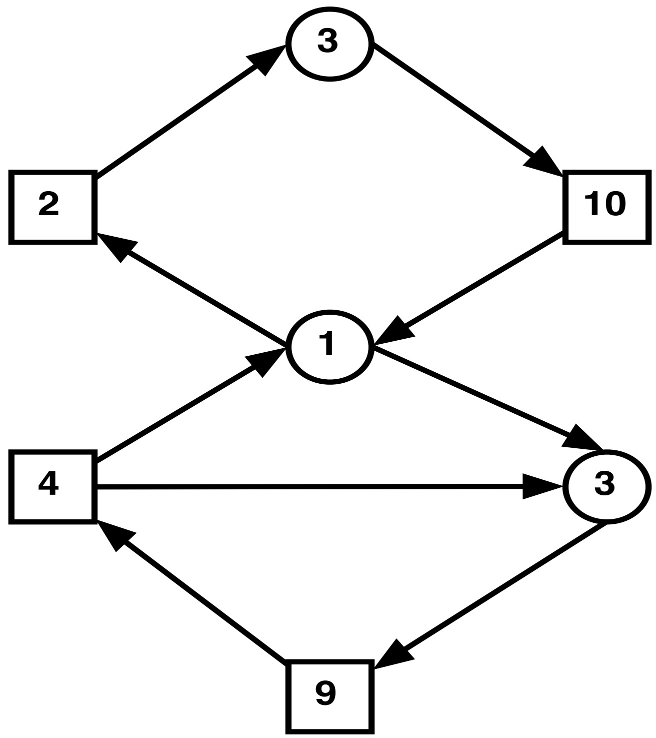
Corrolary

$MF_2(<)$ is decidable

proof: can calculate subset def. by ϕ and decide whether it is A^ω or $\emptyset$

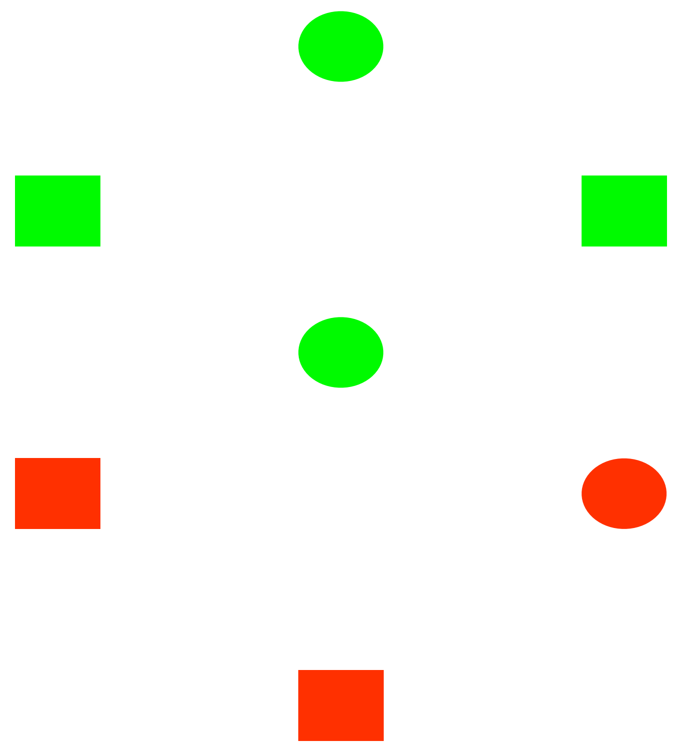
Games and Logic

- **Parity Games**



Proposition

parity games are determined



- **μ-calculus : logic with two fixed points operators**
  - **very common in mathematical practice**
  - **simple way of expressing and checking behavioural properties of computer programs**

ν : least fixed point operator
μ : greatest fixed point operator

$$\phi = \nu y.\mu x.f(x, g(x, y))$$

$\rightarrow$ semantic (vectorial boolean case): models are parity games!

Proposition

each parity game can be characterized by a boolean μ-term that characterizes its winning positions

conversely,

Proposition

the value of a boolean μ-term can be characterized by the winning conditions of a parity game

Corrolary

boolean μ-calculus is decidable

proof: can compute parity game associated with the closed μ-term and since parity game is determined, μ-calculus is decidable.

Conclusion

- **Topics very interconnected**
- **Applications of automata theory and game theory in logic**
- **Algebraic point of view of automata becomes heavy in the infinite case $\rightarrow$ try game theory point of view**