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Limit Knowledge of Rationality

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Introduction

A central concept in epistemic game theory is common knowledge. It is used in basic background assumptions, such as common knowledge of the game structure, or in epistemic hypotheses, such as common knowledge of rationality, that can be employed to epistemically characterize solution concepts. Originally, the notion has been introduced by Lewis (1969) as a prerequisite for a rule to become a convention. Intuitively, some event is regarded as common knowledge among a set of agents, if everyone knows the event, everyone knows that everyone knows the event, everyone knows that everyone knows that everyone knows the event, etc. Following Lewis's (1969) original proposition, it has become standard to define common knowledge as the infinite intersection, or conjunction, of iterated mutual knowledge claims.

A natural question that can be addressed concerns the relationship between the standard definition of common knowledge and the infinite sequence of iterated mutual knowledge underlying it. Indeed, Lipman (1994) considers a specific notion of limit such that common knowledge of the particular event rationality is not equivalent to the limit of iterated mutual knowledge of rationality. Here, a topological approach to set-based epistemic game theory is pursued and it is shown that common knowledge is not equivalent to the topological limit of the sequence of iterated mutual knowledge. On the basis of this observation the new epistemic operator *limit knowledge* is introduced, and some consequences of limit knowledge of the specific event rationality are scrutinized for games.

Results

According to the standard definition, common knowledge of an event is the countably infinite intersection of all successive higher-order mutual knowledge of the event. The existence of situations in which a unique limit point of the sequence of iterated mutual knowledge differs from common knowledge motivates the following definition of the new epistemic operator limit knowledge.

Definition. ▶ *Let $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I})$ be an Aumann structure, $\mathcal{T}$ a topology on $\mathcal{P}(\Omega)$, and E an event. If the limit point of the sequence $(K^m(E))_{m>0}$ is unique, then $LK(E) := \lim_{m\to\infty} K^m(E)$ is the event that E is* limit knowledge *among the set I of agents.* ◀

With limit knowledge, a novel operator is proposed that can be employed for epistemic characterizations of existing or new game-theoretic solution concepts. In this context, situations in which limit knowledge differs from common knowledge are of distinguished interest. It can be shown that such situations necessarily involve sequences of iterated mutual knowledge that are strictly shrinking. A possible application of limit knowledge is given by the following example.

Example. ▶ *Consider the Cournot-type game $\Gamma = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ in normal form with player set $I = \{Alice, Bob, Claire, Donald\}$, strategy sets $S_{Alice} = S_{Bob} = [0,1]$, $S_{Claire} = \{U, D\}$, $S_{Donald} = \{L, R\}$, and utility functions $u_i : S_{Alice} \times S_{Bob} \times S_{Claire} \times S_{Donald} \to \mathbb{R}$ for all $i \in I$ defined as $u_{Alice}(x, y, v, w) = x(1 - x - y)$ and $u_{Bob}(x, y, v, w) = y(1 - x - y)$, and $u_{Claire}(x, y, v, w)$ and $u_{Donald}(x, y, v, w)$ given as follows:*

Claire \ Donald	L	R
U	(2,1)	(1,1)
D	(2,2)	(2,3)

for all $(x, y) \neq (\frac{1}{3}, \frac{1}{3})$

Claire \ Donald	L	R
U	(2,3)	(2,2)
D	(1,1)	(2,1)

for $(x, y) = (\frac{1}{3}, \frac{1}{3})$

Consider also the infinite sequence $(s^n_{Alice}, s^n_{Bob})_{n\geq 0}$ of strategy combinations for Alice and Bob as well as the epistemic model $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i\in I}, (\sigma_i)_{i\in I})$ defined by:

$$(s^n_{Alice}, s^n_{Bob}) = \begin{cases} (0,1) & \text{if } n = 0, \\ (0, \frac{1}{2}) & \text{if } n = 1, \\ (\frac{1-s^{n-1}_{Bob}}{2}, s^{n-1}_{Bob}) & \text{if } n \text{ is even}, \\ (s^{n-1}_{Alice}, \frac{1-s^{n-1}_{Alice}}{2}) & \text{if } n \text{ is odd}, \end{cases}$$

$$\Omega = \{\alpha, \beta, \gamma, \delta, \alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \ldots\},$$

$$\mathcal{I}_{Alice} = \{\{\alpha, \beta, \gamma, \delta\}\} \cup \{\{\alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}, \alpha_{2n+1}, \beta_{2n+1}, \gamma_{2n+1}, \delta_{2n+1}\} : n \geq 0\},$$

$$\mathcal{I}_{Bob} = \{\{\alpha, \beta, \gamma, \delta\}, \{\alpha_0, \beta_0, \gamma_0, \delta_0\}\} \cup \{\{\alpha_{2n-1}, \beta_{2n-1}, \gamma_{2n-1}, \delta_{2n-1}, \alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}\} : n > 0\},$$

$$\mathcal{I}_{Claire} = \{\{\alpha, \beta\}, \{\gamma, \delta\}\} \cup \{\{\alpha_n, \beta_n\} : n \geq 0\} \cup \{\{\gamma_n, \delta_n\} : n \geq 0\},$$

$$\mathcal{I}_{Donald} = \{\{\alpha, \gamma\}, \{\beta, \delta\}\} \cup \{\{\alpha_n, \gamma_n\} : n \geq 0\} \cup \{\{\beta_n, \delta_n\} : n \geq 0\},$$

$$\sigma(\alpha) = (\frac{1}{3}, \frac{1}{3}, U, L), \qquad \sigma(\alpha_n) = (s^n_{Alice}, s^n_{Bob}, U, L),$$

$$\sigma(\beta) = (\frac{1}{3}, \frac{1}{3}, U, R), \qquad \sigma(\beta_n) = (s^n_{Alice}, s^n_{Bob}, U, R),$$

$$\sigma(\gamma) = (\frac{1}{3}, \frac{1}{3}, D, L), \qquad \sigma(\gamma_n) = (s^n_{Alice}, s^n_{Bob}, D, L),$$

$$\sigma(\delta) = (\frac{1}{3}, \frac{1}{3}, D, R), \qquad \sigma(\delta_n) = (s^n_{Alice}, s^n_{Bob}, D, R).$$

Consider finally the topology on $\mathcal{P}(\Omega)$ given by $\{O \subseteq \mathcal{P}(\Omega) : \{\alpha\} \notin O\} \cup \{\mathcal{P}(\Omega)\}$. Then all these definitions imply that first, the sequence $(K^m(R))_{m>0}$ is strictly shrinking, second $CK(R) = \bigcap_{m>0} K^m(R) = \{\alpha, \beta, \gamma, \delta\}$, and third $LK(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = \{\alpha\}$. It follows that $\sigma(CK(R)) = \{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\} = ISD^\Gamma$, while $\sigma(LK(R)) = \{(\frac{1}{3}, \frac{1}{3}, U, L)\} = (ISD + WD)^\Gamma$. Therefore, the solution in accordance with $LK(R)$ is a strict refinement of the solution induced by $CK(R)$. ◀

We now generally show that, for any given game and epistemic model of it satisfying the strictly shrinking condition with respect to iterated mutual knowledge of rationality, every possible event as well as every solution concept can be characterized by limit knowledge of rationality for some appropriate topology.

Theorem. ▶ *Let Γ be a normal form and $\mathcal{A}^\Gamma$ an epistemic model of it such that $(K^m(R))_{m>0}$ is strictly shrinking.*

1. *Let E be an event. Then, there exists a topology on $\mathcal{P}(\Omega)$ such that $LK(R) = E$.*
2. *Let $\mathcal{SC}$ be a solution concept. Then, there exists a topology on $\mathcal{P}(\Omega)$ such that $\sigma(LK(R)) \subseteq \mathcal{SC}^\Gamma$.* ◀

Epistemic hypotheses being particular events, the above theorem shows that limit knowledge of rationality can be used as a topological foundation for any epistemic hypothesis as well as an epistemic-topological foundation for any solution concept. Yet note that this universal characterization capability of limit knowledge of rationality indispensably requires the strictly shrinking condition to hold. Hence, the expressive power of this epistemic operator is somewhat countered by this significant constraint.

Moreover, observe that the above theorem can be refined towards equality in the sense that for any epistemic model $\mathcal{A}^\Gamma$ fulfilling its assumptions as well as the additional condition $\sigma(\Omega) \supseteq \mathcal{SC}^\Gamma$, there exists a topology such that $\sigma(LK(R)) = \mathcal{SC}^\Gamma$. In this case, limit knowledge of rationality thus provides an exact epistemic-topological foundation for the given solution concept.

Finally, the proof of the theorem actually provides a generic method to construct a topology such that $\lim_{m\to\infty}(K^m(R))_{m>0} = \sigma^{-1}(\mathcal{SC}^\Gamma)$. The definition of this topology is completely independent from the specific game considered. However, the convergence properties of the sequence $(K^m(R))_{m>0}$ according to this topology do depend on the underlying game. Consequently, the well-definedness and characterization capability of limit knowledge of rationality do also depend on the underlying game.

Discussion

Limit knowledge can be understood as the event which is approached by the sequence of iterated mutual knowledge, according to some notion of closeness between events. Moreover, likewise other epistemic hypotheses, limit knowledge of rationality can also be associated with a kind of reasoning pattern of the agents. Indeed, by definition $LK(R) = \lim_{m\to\infty} K^m(R)$, hence it follows that $LK(R)$ holds i.e. the actual world ω belongs to $LK(R)$, if and only if there exists some event E such that both $\omega \in E$ and $E = \lim_{m\to\infty} K^m(R)$, meaning that everyone considers possible a true event which is the topological limit of the sequence $(K^m(R))_{m>0}$. Hence $\omega \in LK(R)$ can be interpreted as everyone considering possible a true event which is eventually topologically indistinguishable from all remaining higher-order mutual knowledge of rationality.

The main theorem ensures that several implications of limit knowledge of rationality for epistemic hypotheses as well as for solution concepts in games could be relevant. This epistemic-topological insight can be apprehended from two different angles. A first approach would study possible topological characterizations via limit knowledge of rationality for a given epistemic hypothesis or solution concept. Relevant topological reasoning patterns of the agents in accordance with some given epistemic hypothesis or solution concept could thus be unveiled.

A second approach would derive the epistemic hypotheses or solution concepts in accordance with limit knowledge of rationality, for some given topology. It might be of particular interest to explore the game-theoretic consequences of topologies being defined on the basis of relevant descriptions of the event space or revealing cogent underlying reasoning patterns of the agents. Such topologies could be called *epistemically plausible*. Solution concepts characterizable in this way might be argued to gain in credibility compared to ones that are not. An instance of such an epistemically plausible topological foundation for the solution concept n-times strict dominance in pure strategies SD^n is given in the full paper.

Finally, it is envisioned to construct a general topological framework for Aumann structures to enrich the epistemic analysis of games. Such an approach could, for instance, be capable of phrasing and reflecting the epistemic properties of an interactive situation in topological terms.

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