The Algebraic Counterpart of the Wagner Hierarchy

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Outline

1. Introduction
2. ω-Automata
3. The Wagner hierarchy
4. ω-Semigroups
5. The $\mathbb{FSG}$-hierarchy
6. Conclusion

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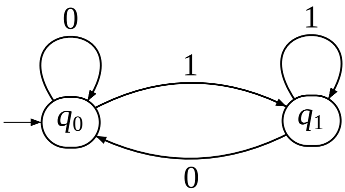
Classifying ω-rational languages

Automata
The Wagner Hierarchy
[Wag79]

Algebra
[CarPer96,99], [DupRis06]
The FSG-hierarchy

Muller automaton (deterministic)

$\mathcal{A}$ is a labeled graph, with $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{P}^{Q}$.



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Then $L^{\omega}(\mathcal{A}) = A^{*}(0^{\omega} \cup 1^{\omega})$.

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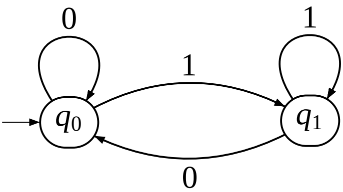

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Then $L^{\omega}(\mathcal{A}) = A^{*}(0^{\omega} \cup 1^{\omega})$.

Theorem

The following conditions are equivalent:

1. L *is* ω*-rational*
2. L *is recognizable by a finite Muller automaton.*

The Wagner Hierarchy

(classification of ω-rational languages)

Definition (Wagner hierarchy)

The collection of all ω-rational languages ordered by the continuous reduction ($\leq_W$) is called *the Wagner hierarchy*.

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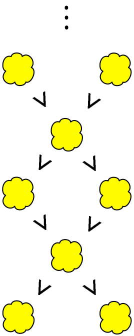

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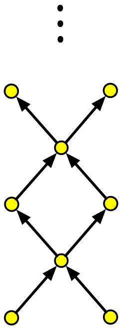

Theorem

The Wagner hierarchy has a height of ω^ω, and it is decidable.

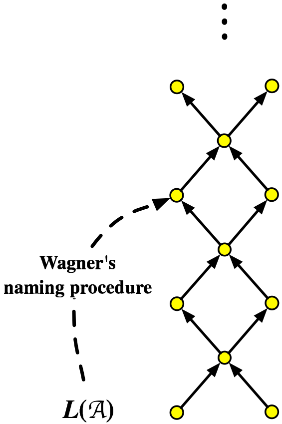

Illustration of Wagner's decidability procedure

Classifying ω-rational language is equivalent to classifying their underlying Muller automata.

$$\mathcal{T} = \{\{q_0\}, \{q_2, q_3\}, \{q_4\}, \{q_6, q_7\}, \{q_8\}, \{q_{10}, q_{11}\}, \{q_{12}\}\}$$

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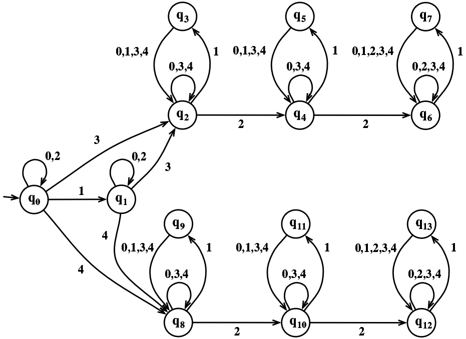

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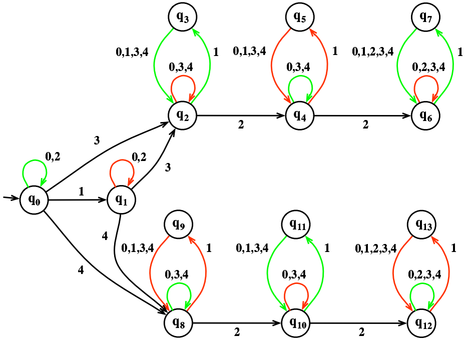

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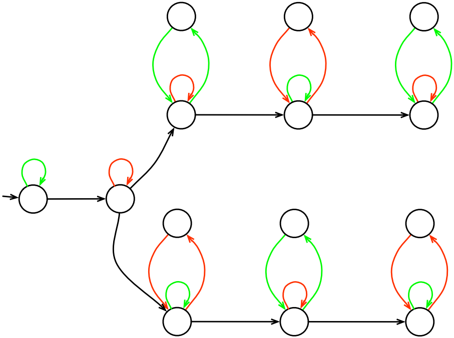

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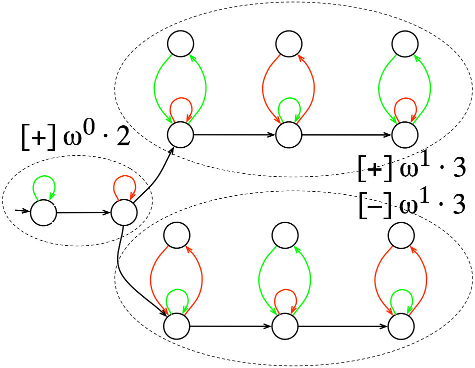

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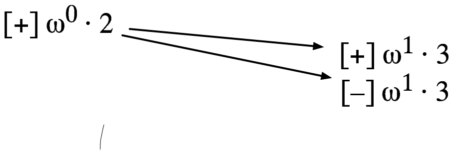

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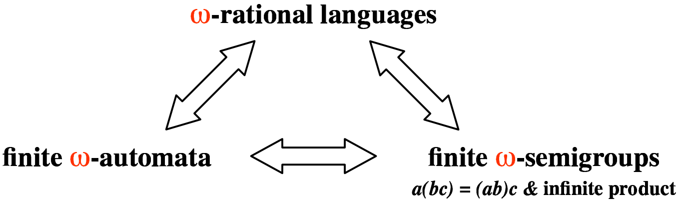
$$\begin{tikzcd} & {[+]\,\omega^1 \cdot 3} \\ {[+]\,\omega^0 \cdot 2} \arrow[ur] \arrow[r] & {[-]\,\omega^1 \cdot 3} \end{tikzcd}$$

$$d_W(\mathcal{A}) = d_W(L(\mathcal{A})) = \omega \cdot 3 + 2$$

and $L(\mathcal{A})$ is non-self-dual

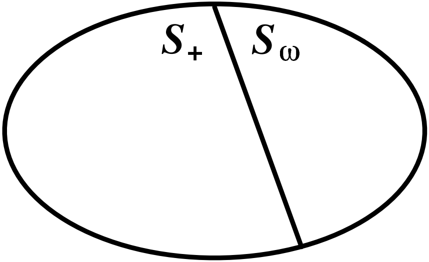
$$\mathcal{T} = \{\{q_0\}, \{q_2, q_3\}, \{q_4\}, \{q_6, q_7\}, \{q_8\}, \{q_{10}, q_{11}\}, \{q_{12}\}\}$$


ω-rational languages
finite ω-automata
finite ω-semigroups
a(bc) = (ab)c & infinite product

An ω-semigroup is a semigroup equipped with a suitable infinite product.

ω-semigroup

$$S = (S_+, S_\omega)$$



Notation: $\pi_S(x, x, x, x, \ldots)$ will be written x^ω.

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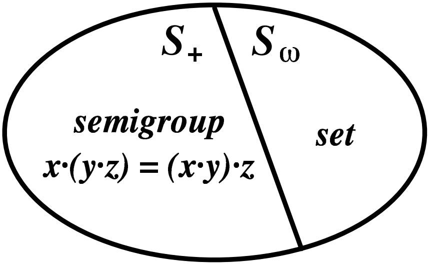

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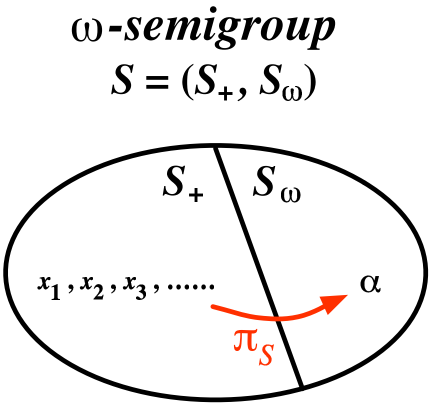

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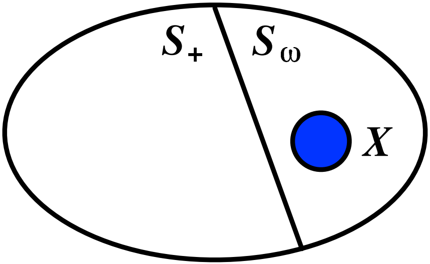
pointed* ω*-semigroup

$$\left(S = (S_+, S_\omega),\ X \subseteq S_\omega \right)$$



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Finite pointed ω-semigroups are the algebraic counterparts of Muller automata.

Theorem

An ω-language is recognizable by a finite pointed ω-semigroup iff it is recognizable by a finite Muller automaton (iff it is ω-rational).

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Example

Consider the language $K = ((a+b)^* a)^\omega$. Then $Synt(K) = (S, X)$, where

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 0 \qquad 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1$$
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- $X = \{0^\omega\}$.

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Example

Consider the language $L = (a\{b,c\}^* \cup \{b\})^\omega$. Then $Synt(L) = (T, Y)$, where

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

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\begin{array}{cccc}
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We aim to classify finite pointed ω-semigroups. We adopt a hierarchical game approach.

Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let also $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.

The infinite two-player game $\mathbb{SG}\left((S, X), (T, Y)\right)$ is defined as follows:

- Player I plays elements from S_+,
- Player II plays elements from T_+,
- players I and II play alternately, Player I begins,
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\mathrm{I}: & s_0 & & & s_1 & & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (s_0, s_1, s_2, \cdots) \\
 & & \searrow & \nearrow & & \searrow & & & \\
\mathrm{II}: & & & - & & t_0 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (t_0, t_1, t_2, \cdots)
\end{array}
$$

- Player II wins the game $\mathbb{SG}((S, X), (T, Y))$ iff

$$
\begin{array}{lll}
\text{either} & \pi_S(s_0, s_1, \ldots) \in X \text{ and } \pi_T(t_0, t_1, \ldots) \in Y \\
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Definition ($\mathbb{SG}$-reduction)

We write $(S, X) \leq_{SG} (T, Y)$ iff Player II has a winning strategy in $\mathbb{SG}((S, X), (T, Y))$. And as usual, $(S, X) <_{SG} (T, Y)$ iff $(S, X) \leq_{SG} (T, Y) \not\leq_{SG} (S, X)$; also $(S, X) \equiv_{SG} (T, Y)$ iff $(S, X) \leq_{SG} (T, Y) \leq_{SG} (S, X)$.

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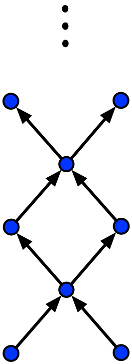
Definition ($\mathbb{FSG}$-hierarchy)

The collection of finite pointed ω-semigroups ordered by the $\leq_{SG}$-relation is called *the* $\mathbb{FSG}$*-hierarchy*.



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Theorem

The $\mathbb{FSG}$-hierarchy and the Wagner hierarchy are isomorphic.

Corollary

The $\mathbb{FSG}$-hierarchy has a height of ω^ω, and it is decidable.

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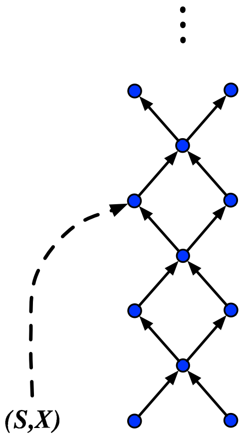
We present a strictly algebraic decidability procedure of the $\mathbb{FSG}$-hierarchy.



Example

Consider the syntactic pointed ω-semigroup of $K = ((a+b)^*a)^\omega$:

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

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- $X = \{0^\omega\}$.

Example (continued)

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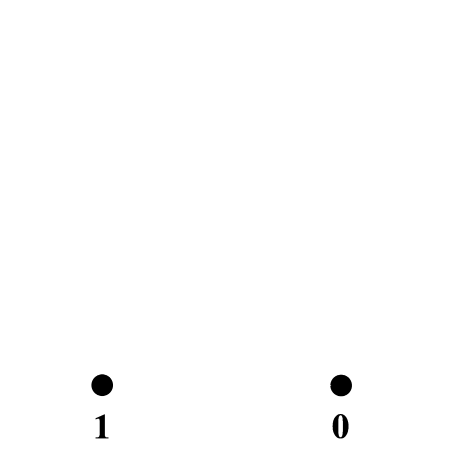


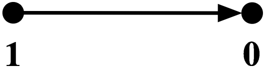
Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

$$\begin{array}{ll} 0 \cdot 0 = 0 & 0 \cdot 1 = 0 \\ 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega \\ 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$

- $X = \{0^\omega\}$.



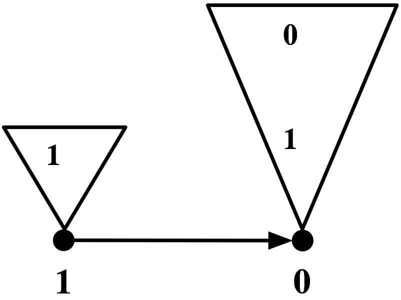
Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

$$0 \cdot 0 = 0 \quad 0 \cdot 1 = 0$$
$$1 \cdot 0 = 0 \quad 1 \cdot 1 = 1$$
$$00^\omega = 0^\omega \quad 10^\omega = 0^\omega$$
$$01^\omega = 1^\omega \quad 11^\omega = 1^\omega$$

- $X = \{0^\omega\}$.



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Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

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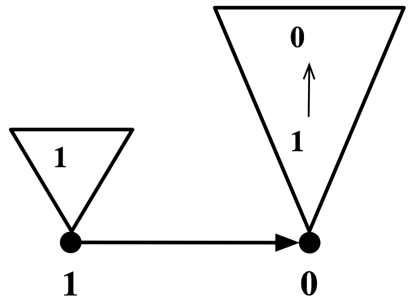
- $X = \{0^\omega\}$.



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Example (continued)

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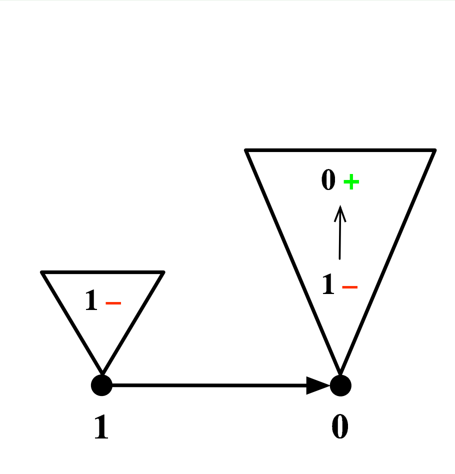

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Example (continued)

- $S = (\{0, 1\}, \{0^\omega, 1^\omega\})$

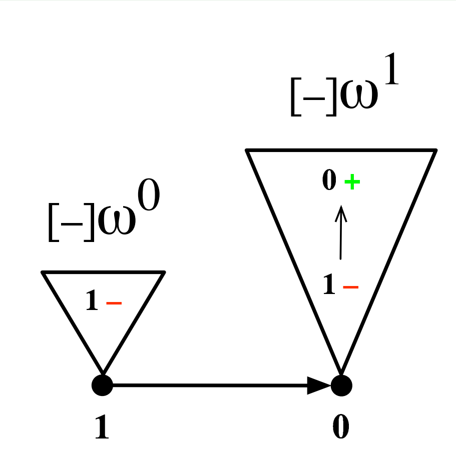
  $0 \cdot 0 = 0 \quad 0 \cdot 1 = 0$

  $1 \cdot 0 = 0 \quad 1 \cdot 1 = 1$

  $00^\omega = 0^\omega \quad 10^\omega = 0^\omega$

  $01^\omega = 1^\omega \quad 11^\omega = 1^\omega$

- $X = \{0^\omega\}$.



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Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

  $0 \cdot 0 = 0 \quad 0 \cdot 1 = 0$

  $1 \cdot 0 = 0 \quad 1 \cdot 1 = 1$

  $00^\omega = 0^\omega \quad 10^\omega = 0^\omega$

  $01^\omega = 1^\omega \quad 11^\omega = 1^\omega$

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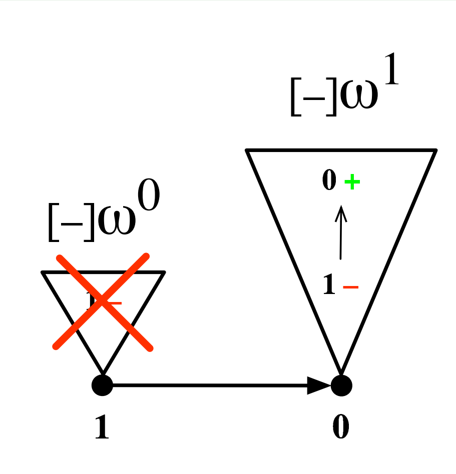

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Example (continued)

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

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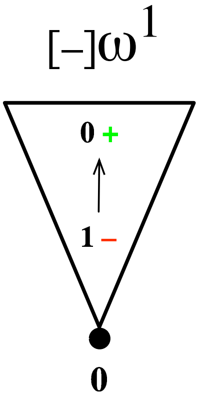

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- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$

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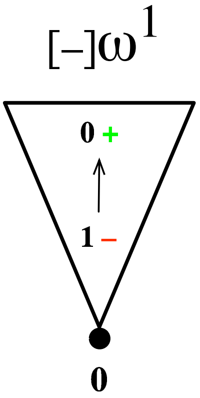


Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example

Consider the syntactic pointed ω-semigroup of $L = (a\{b,c\}^* \cup \{b\})^\omega$:

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

$$
\begin{array}{cccc}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega
\end{array}
$$

- $Y = \{a^\omega\}$.

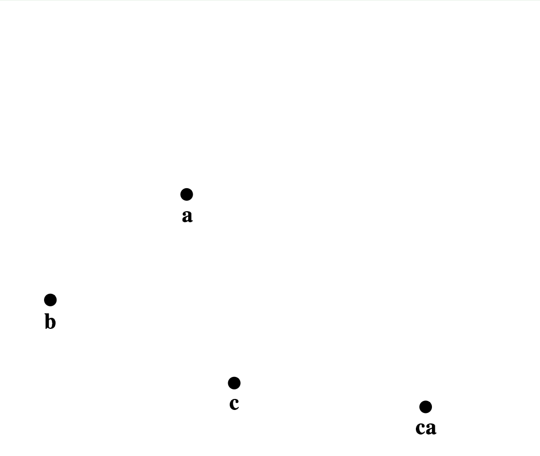
Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

$a^2 = ab = ac = ba = a \qquad b^2 = b \qquad bc = cb = c^2 = c$

$b^\omega = aa^\omega = a(ca)^\omega = ba^\omega = a^\omega \qquad c^\omega = 0 \qquad b(ca)^\omega = ca^\omega = c(ca)^\omega = (ca)^\omega$

- $Y = \{a^\omega\}$.



Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

  $a^2 = ab = ac = ba = a$ $\quad b^2 = b$ $\quad bc = cb = c^2 = c$

  $b^\omega = aa^\omega = a(ca)^\omega = ba^\omega = a^\omega$ $\quad c^\omega = 0$ $\quad b(ca)^\omega = ca^\omega = c(ca)^\omega = (ca)^\omega$

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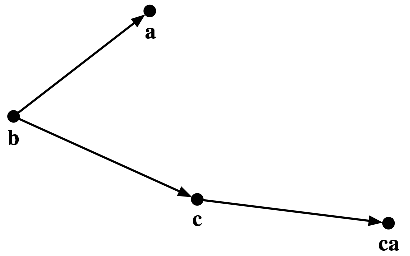

Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

$$a^2 = ab = ac = ba = a \qquad b^2 = b \qquad bc = cb = c^2 = c$$
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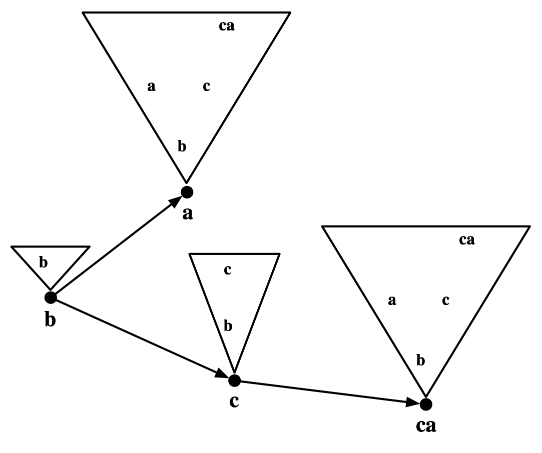

Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

  $a^2 = ab = ac = ba = a \quad b^2 = b \quad bc = cb = c^2 = c$

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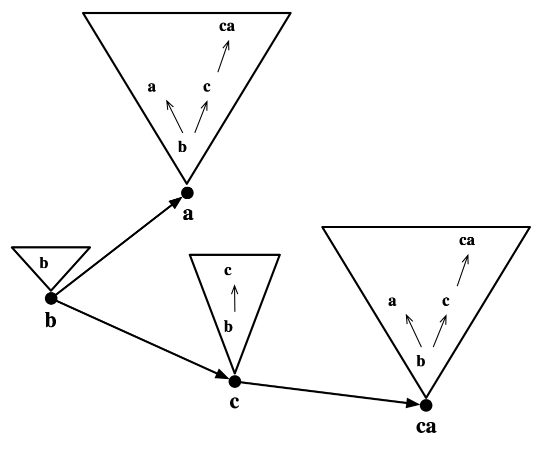

Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

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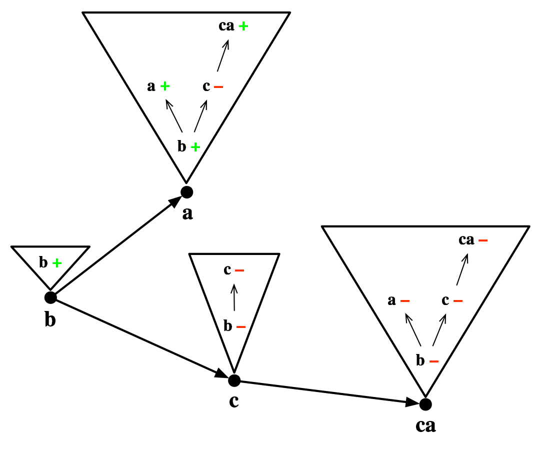

Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

  $a^2 = ab = ac = ba = a \quad b^2 = b \quad bc = cb = c^2 = c$

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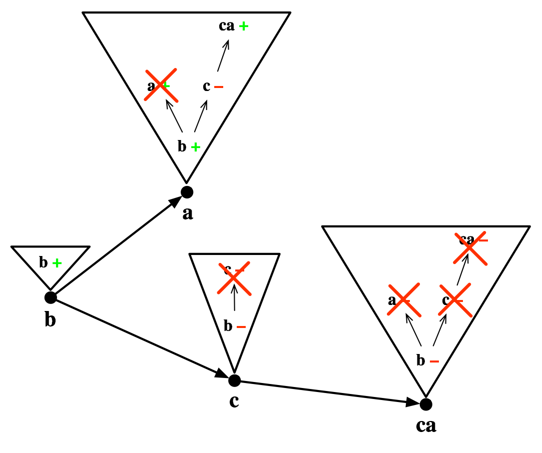

Example (continued)

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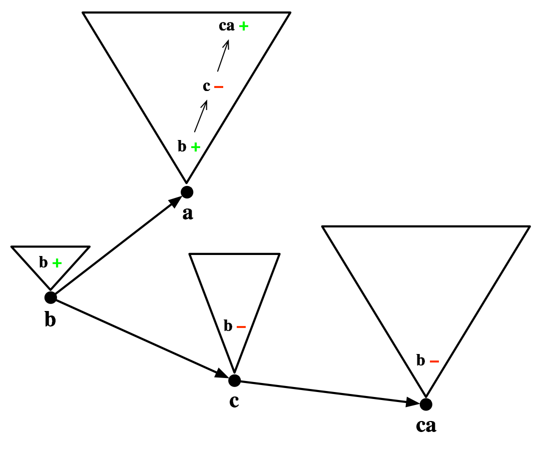

Example (continued)

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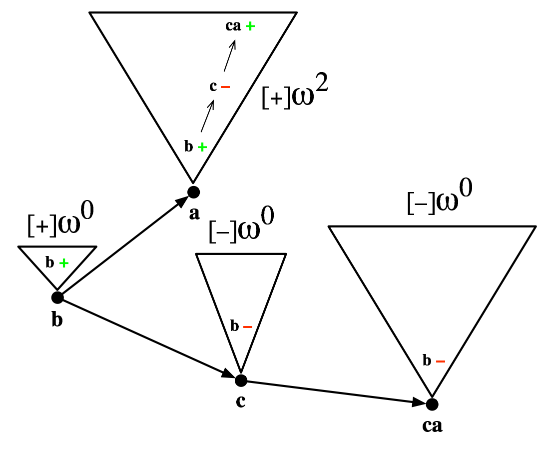

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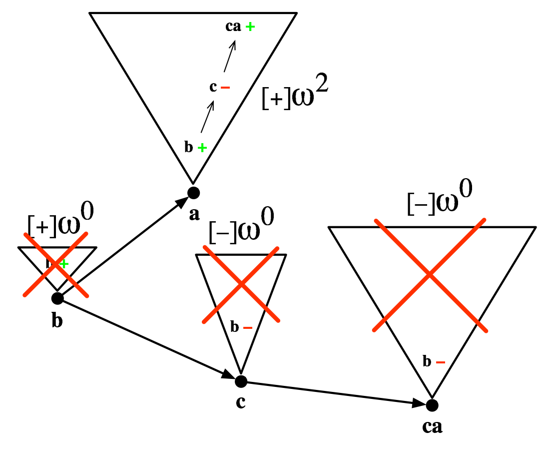

Example (continued)

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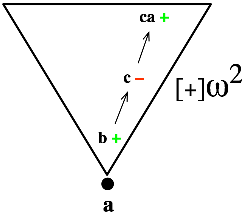

Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$

$a^2 = ab = ac = ba = a \quad b^2 = b \quad bc = cb = c^2 = c$

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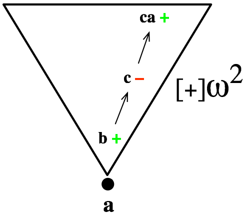


Therefore $d_{SG}((T,Y)) = d_W(L) = \omega^2$, and L is non-self-dual.

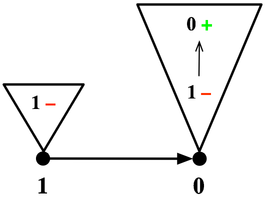
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

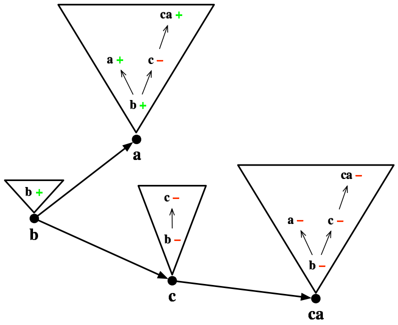
(S,X) I

(T,Y) II

Player II

Player I





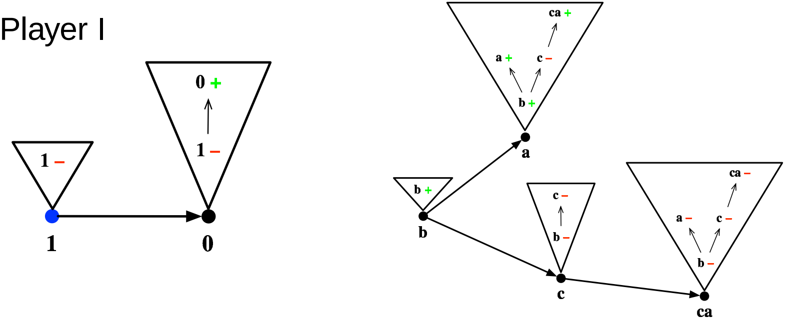
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X) I 1

(T,Y) II

Player II

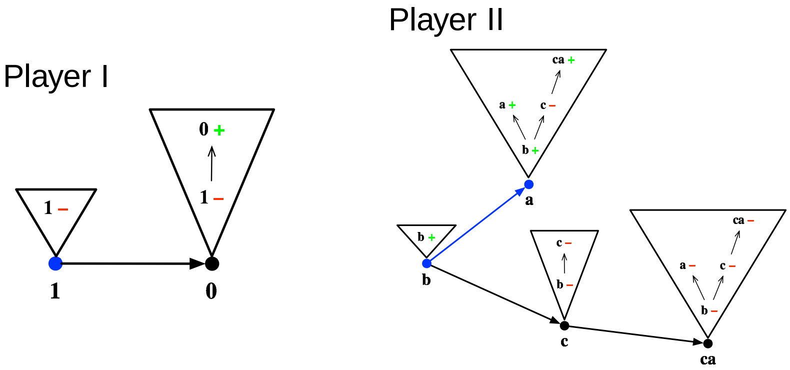
Player I



Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X) **I** **1**

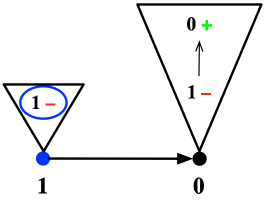
(T,Y) **II** ***a***



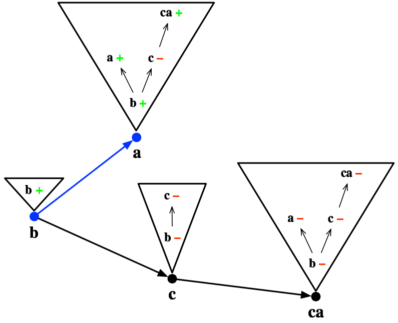
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	

Player I



Player II

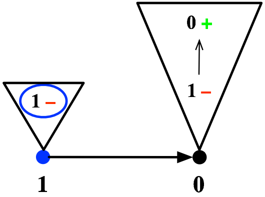


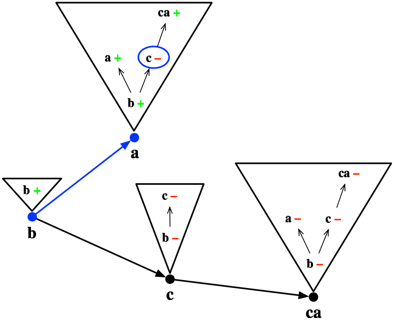
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

Player I





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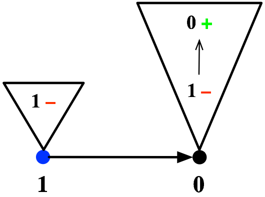
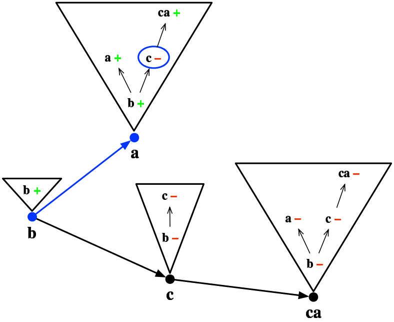
(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

Player I





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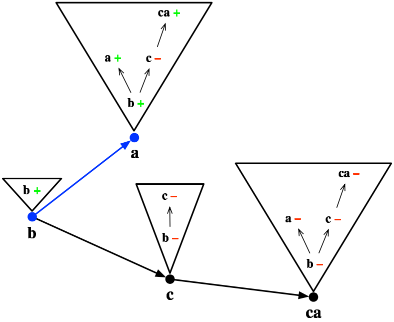
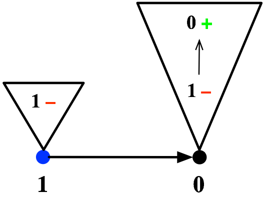
(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

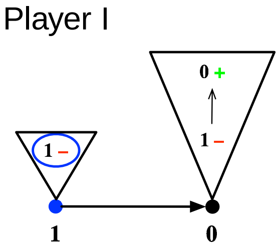


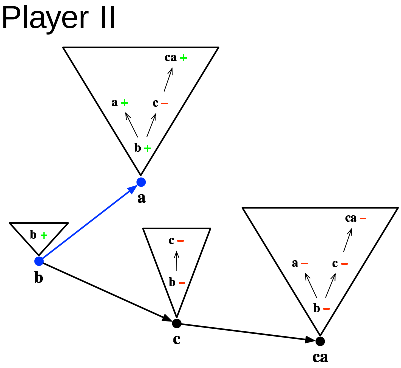
Player I



Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

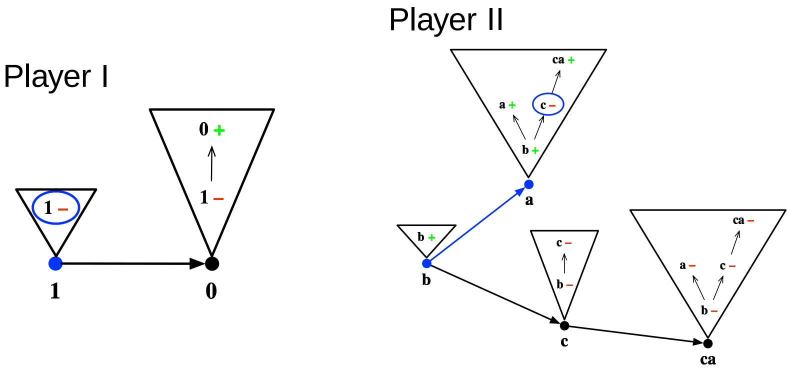
(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	





Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***



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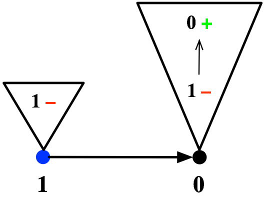
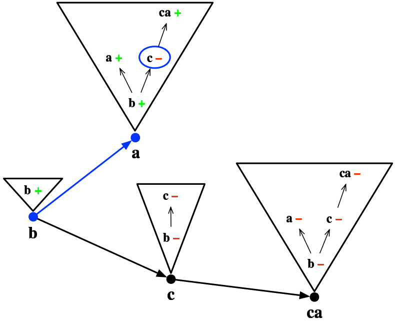
(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player I



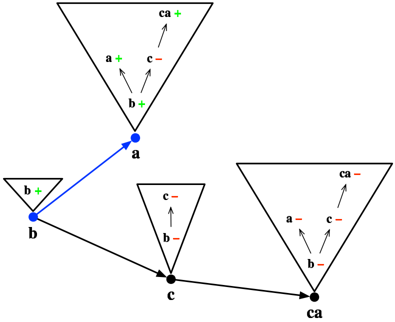
Player II



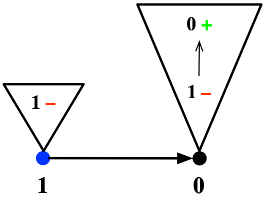
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player II



Player I

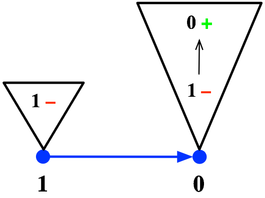


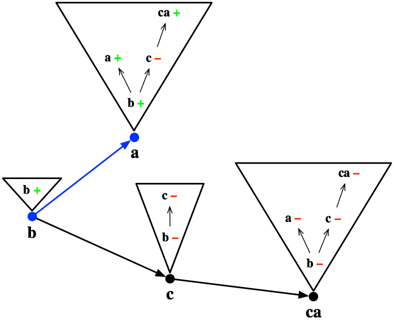
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	

Player II

Player I

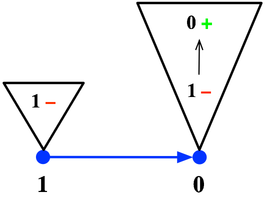




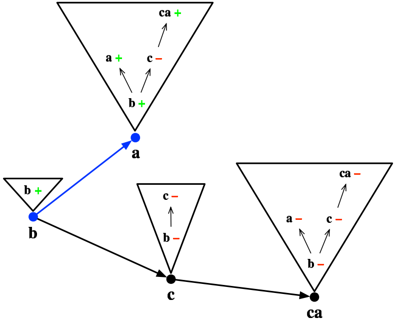
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**

Player I



Player II

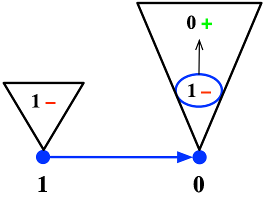


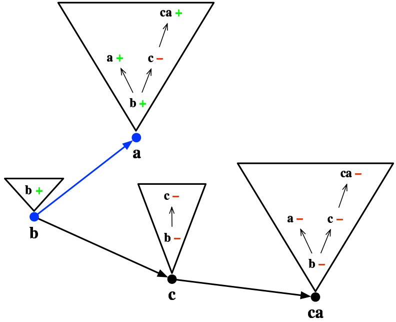
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	

Player II

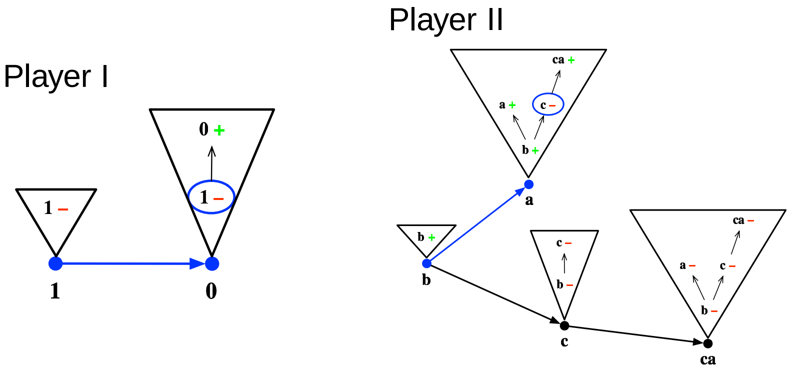
Player I





Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

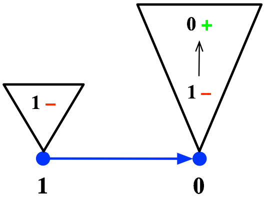
(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***



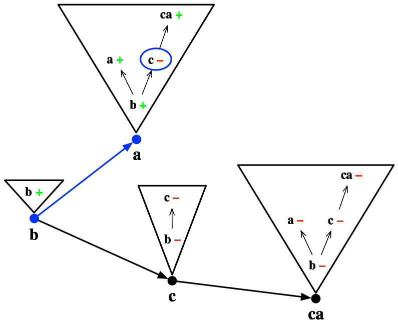
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

Player I



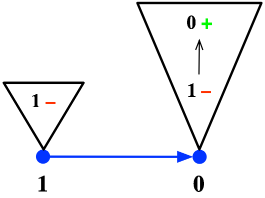
Player II



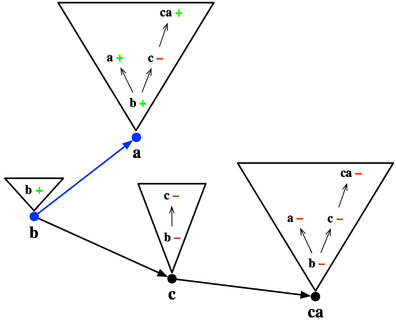
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

Player I



Player II

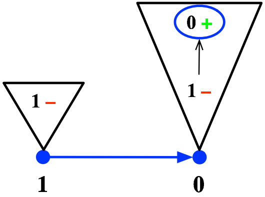


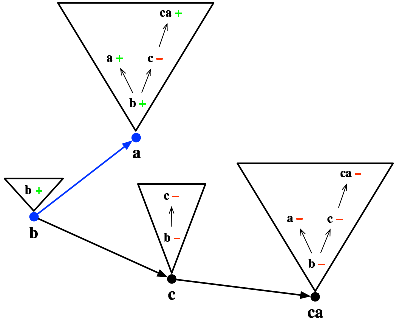
Therefore $(S,X) \leq_{SG} (T,Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	

Player II

Player I



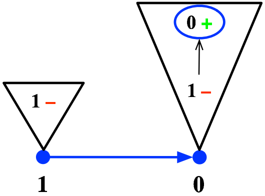


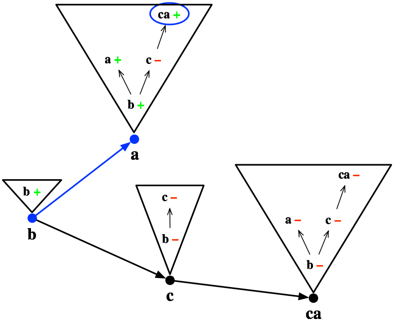
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player II

Player I

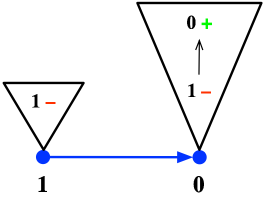




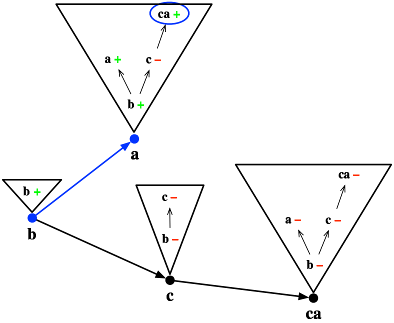
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player I



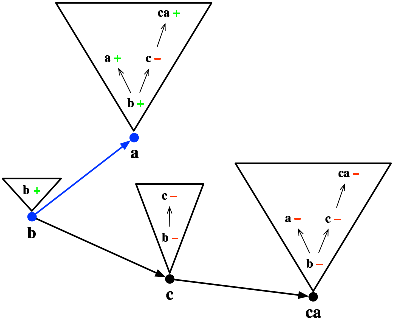
Player II



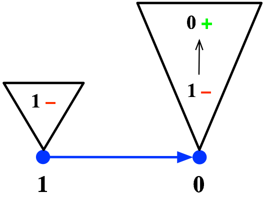
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player II



Player I

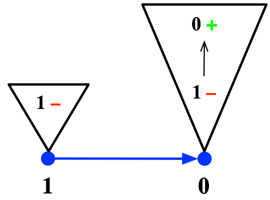


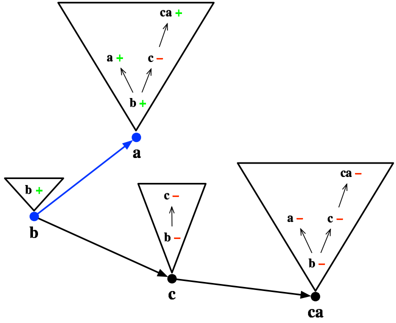
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**	**……**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***	**……**

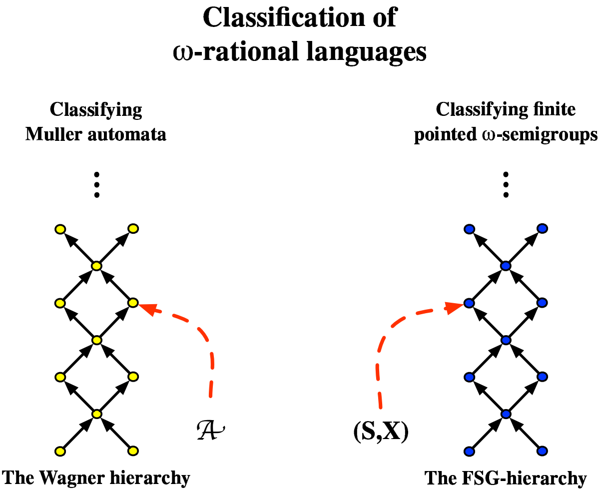
Player II

Player I





Summary


Classification of
ω-rational languages
Classifying
Muller automata
Classifying finite
pointed ω-semigroups
$\mathcal{A}$
(S,X)
The Wagner hierarchy
The FSG-hierarchy