Analog Recurrent Neural Networks on Infinite Input Streams

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ω-automata

ω-ARNNs

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Theorem (Siegelmann & Sontag)

- *Analog recurrent neural networks (ARNNs) can recognize in exponential time all possible languages.*
- *Analog recurrent neural networks recognize in polynomial time the class of languages $P/poly$.*

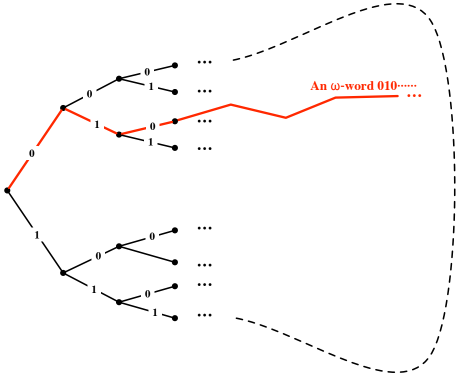
We prove the counterpart of this result in the context of infinite computation.

Definition

The *Cantor space* $\mathcal{C} = \{0,1\}^\omega$ is the set of all infinite sequences of bits.

For instance, $010110001010001010 0 \cdots \in \mathcal{C}$

The Cantor space



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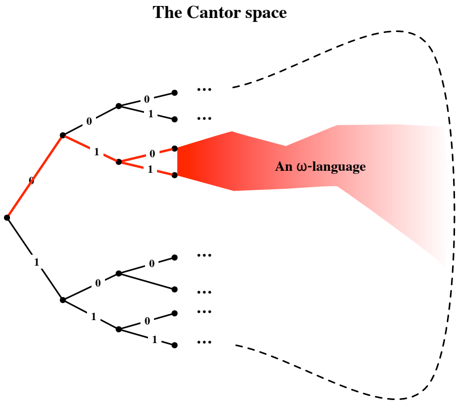
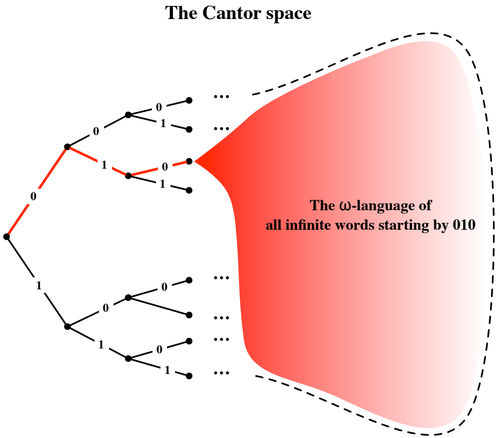


We equip the Cantor space with the product topology of the discrete topology on $\{0, 1\}$.

The basic open ω-languages are of the form $L = \mathit{prefix} \cdot \{0, 1\}^\omega$



Definition

The levels of the *Borel hierarchy* are defined by induction on $\alpha < \omega_1$ as follows:

- $\mathbf{\Sigma}^0_1 = \{A \subseteq \mathcal{C} : A \text{ is open}\}$
- $\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n \in \mathbb{N}} A_n : A_n \in \mathbf{\Pi}^0_\beta \text{ for } \beta < \alpha\}$
- $\mathbf{\Pi}^0_\alpha = \{A : A^\complement \in \mathbf{\Sigma}^0_\alpha\}$
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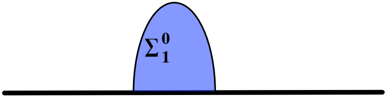

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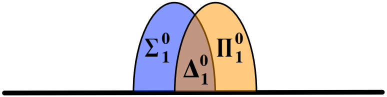

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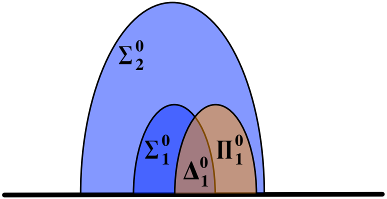

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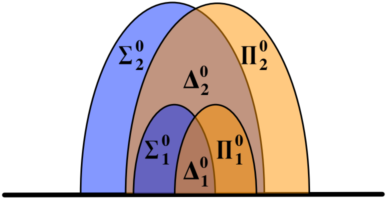

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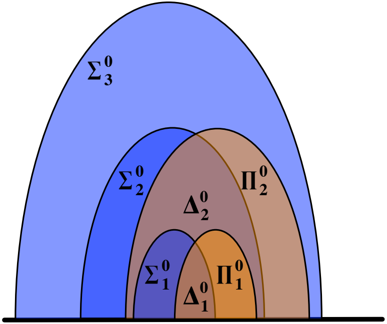

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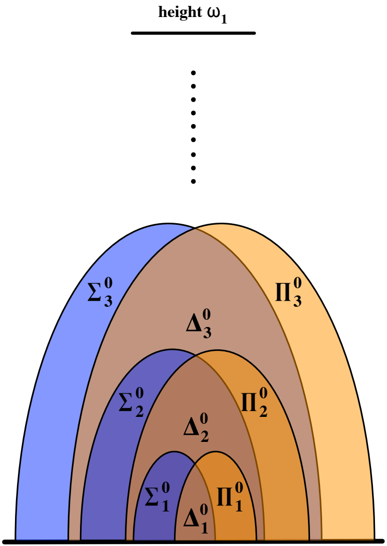

The Borel Hierarchy

height ω_1

$$\begin{array}{ccc}
 & \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
 & \subsetneq \quad \subsetneq & \\
 & \boldsymbol{\Delta}^0_3 & \\
 & \subsetneq \quad \subsetneq & \\
\boldsymbol{\Sigma}^0_2 & & \boldsymbol{\Pi}^0_2 \\
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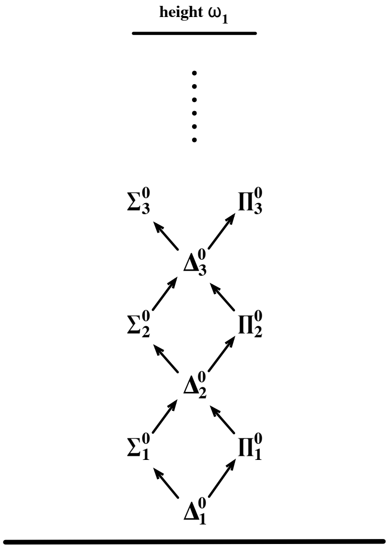
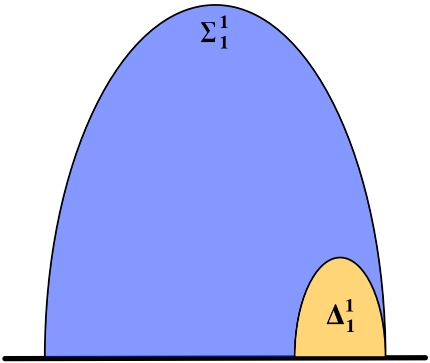


Definition

An ω-language is called *analytic* ($\mathbf{\Sigma}^1_1$) if it is the 1st projection of some $\mathbf{\Pi}^0_2$-set of $\mathcal{C} \times \mathcal{C}$.

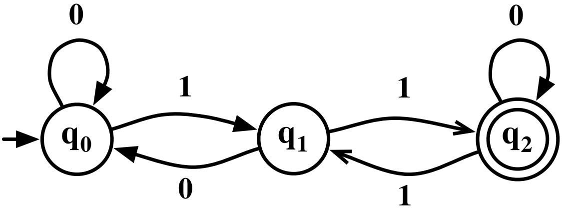
The class of analytic sets strictly contains the class of Borel sets $\mathbf{\Sigma}^1_1 \supsetneq \mathbf{\Delta}^1_1$



Computation over infinite objects

- Purely theoretical interest
- Connections with formal language theory, game theory, algebra, and logics
- Practical implications: modeling infinite behavior, verification of non-terminating reactive systems, ...

Classical finite automaton



The set of words accepted by the automaton is the called the language recognized by this automaton.

Many other kinds of abstract machines: counter automata, pushdown automata, Turing machines, etc.

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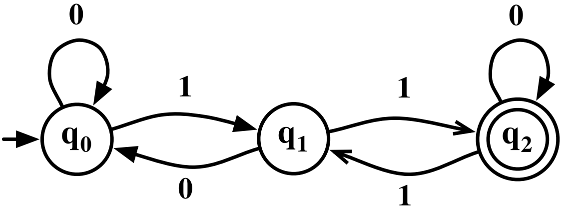

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ω-automaton (Muller)

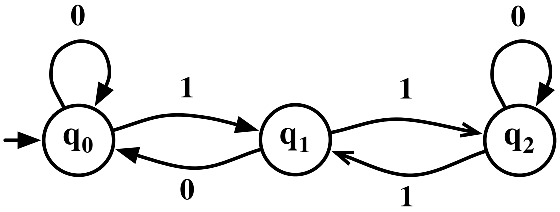

table $\mathcal{T} = \{\{q_1, q_2\}, \{q_2\}\}$

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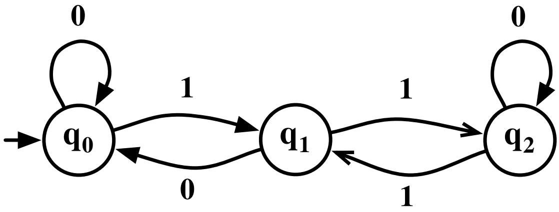

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input 001111111111 . . . . . . accepted

input 010101010101 . . . . . . rejected

input 110000000000 . . . . . . accepted

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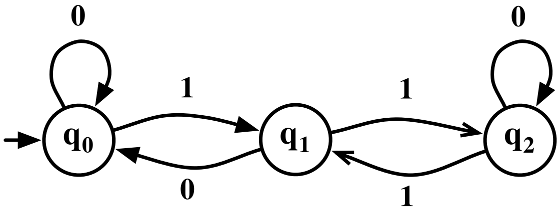

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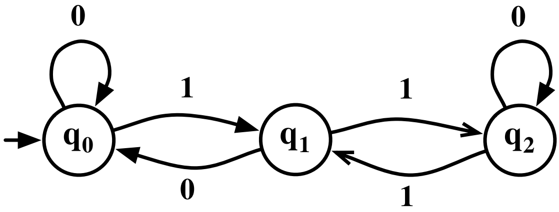

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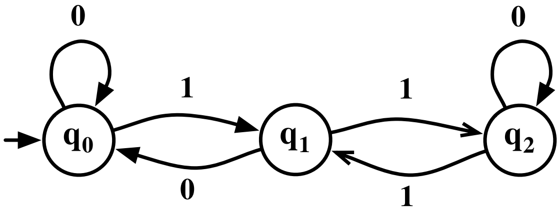

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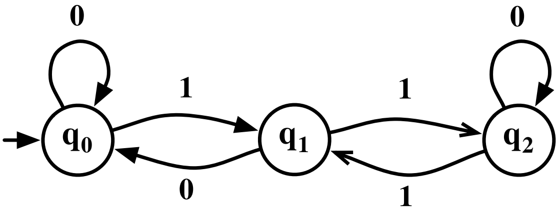

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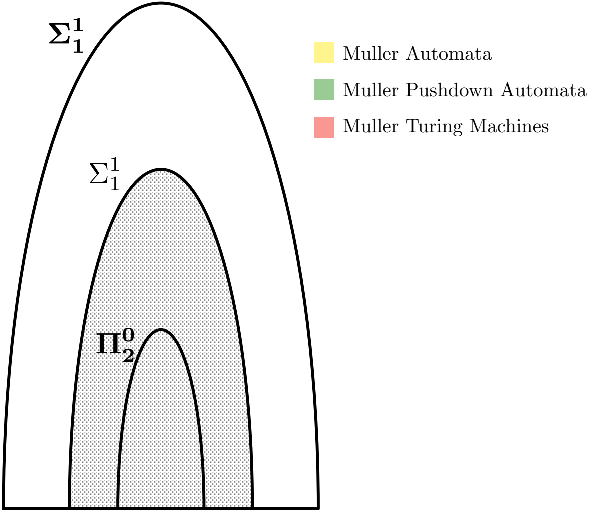
Many other kinds of abstract ω-machines: Muller counter automata, Muller pushdown automata, Muller Turing machines, etc.

Deterministic case



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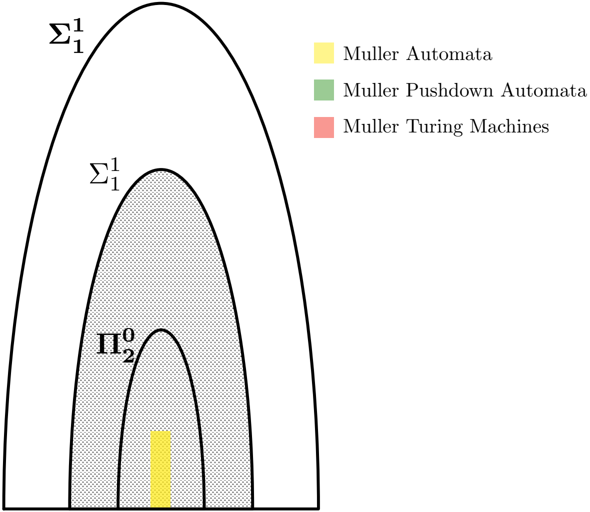

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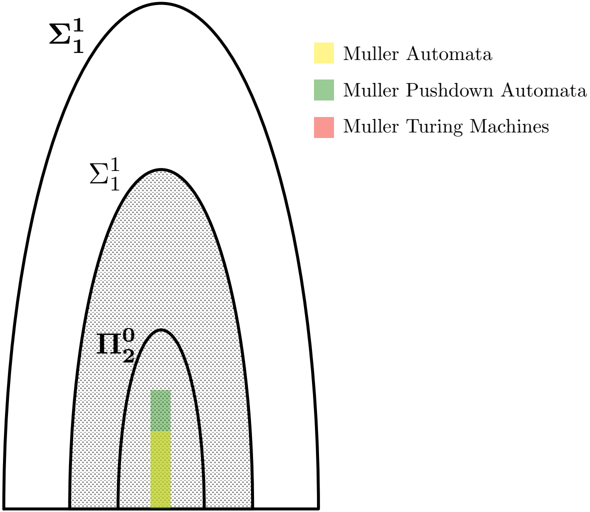

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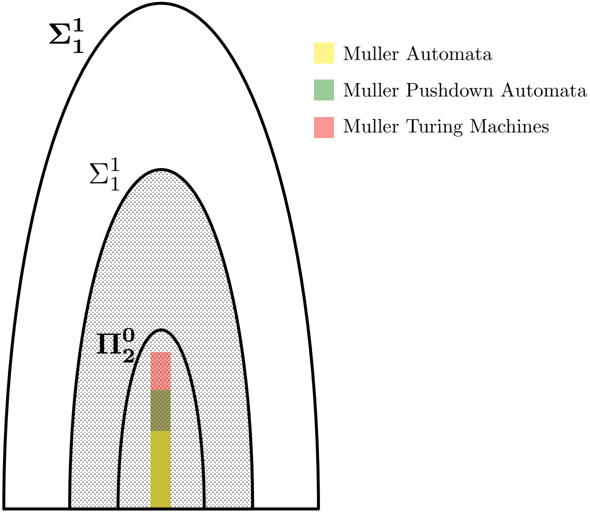
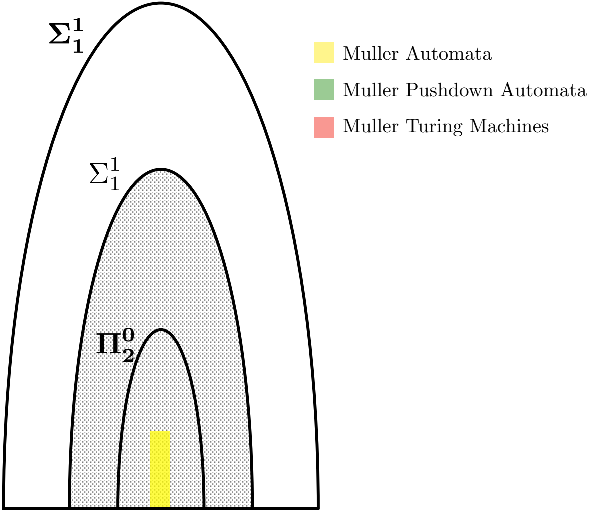


Non-deterministic case



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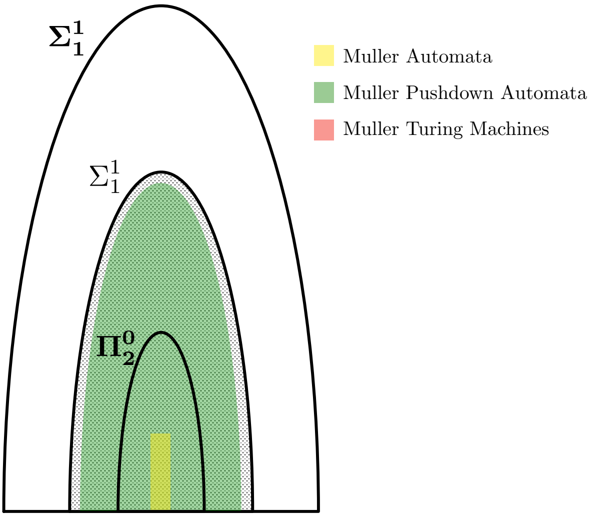

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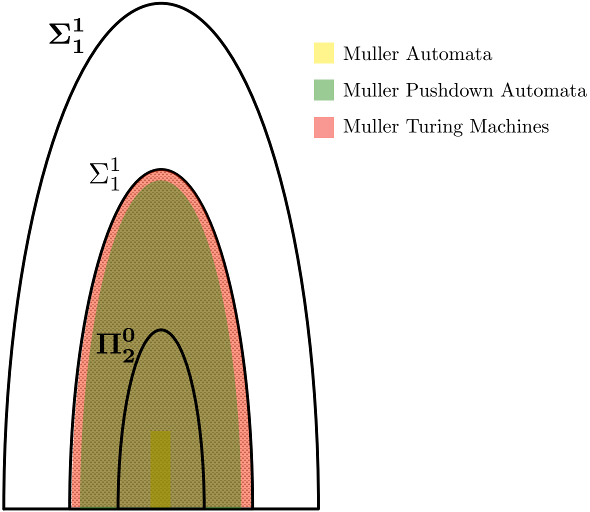
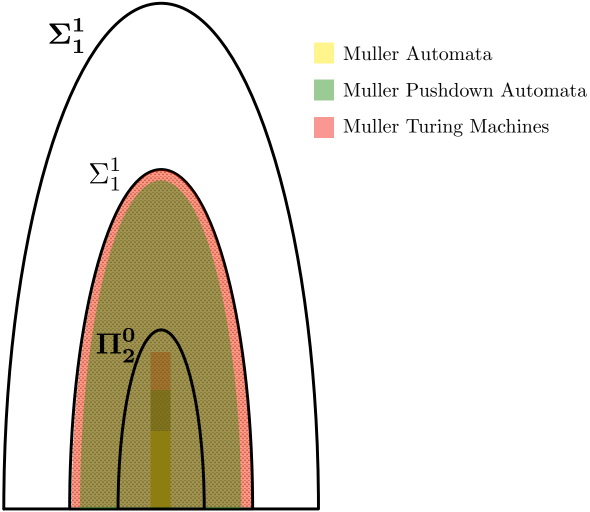
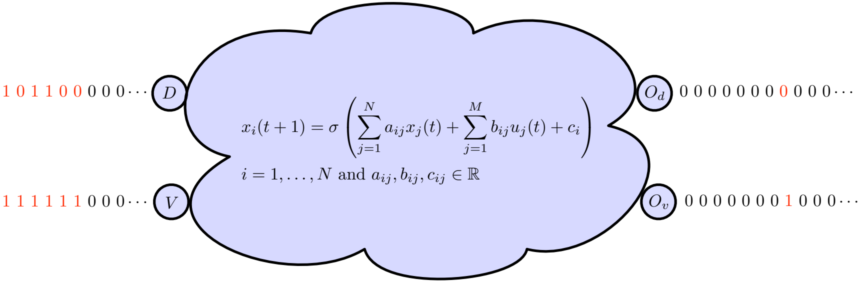


Deterministic and non-deterministic cases

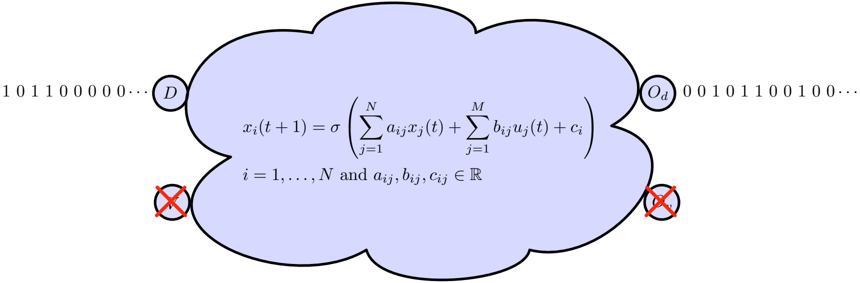


Classical ARNNs



The set of finite words accepted by $\mathcal{N}$ is called the language recognized by $\mathcal{N}$.

ω-ARNNs



An ω-word is said to be accepted by $\mathcal{N}$ iff it induces infinitely many output answers (spikes) along the infinite computation.

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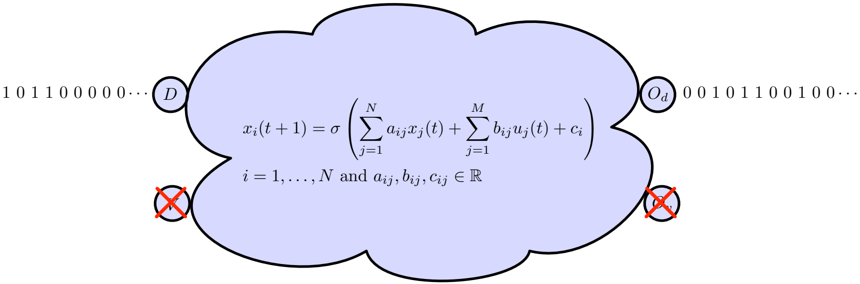
ω-ARNNs



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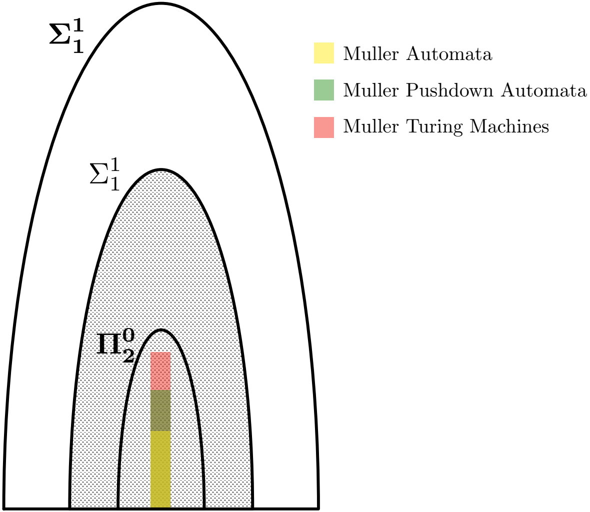

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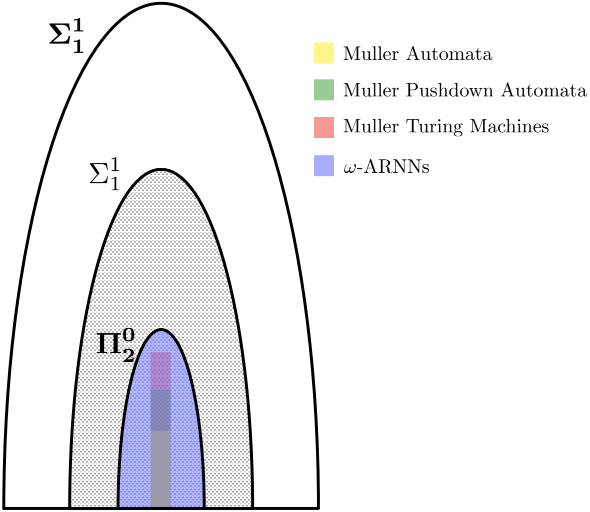

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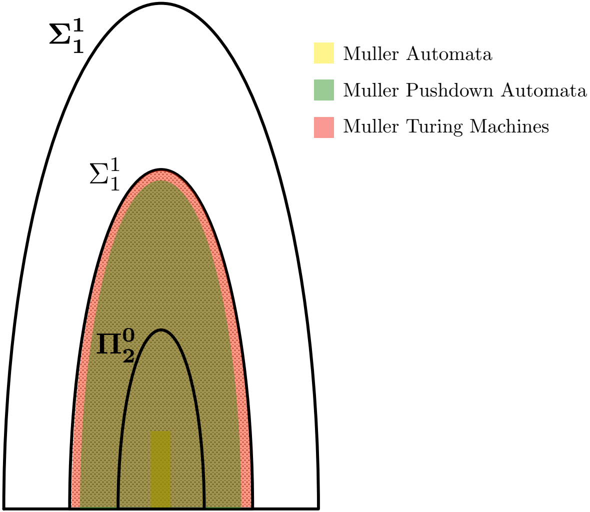

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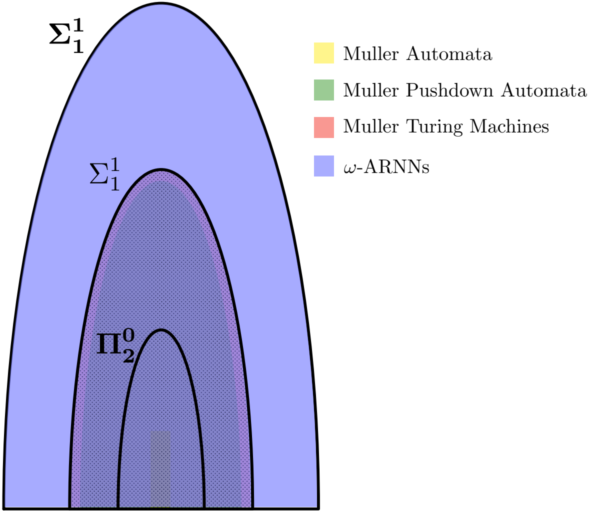

Theorem

Let $L \subseteq \{0,1\}^\omega$ be an ω-language. The following conditions are equivalent:

1. *L is recognizable by some deterministic ω-ARNN*
2. $L \in \mathbf{\Pi}^0_2$

Theorem

Let $L \subseteq \{0,1\}^\omega$ be an ω-language. The following conditions are equivalent:

1. *L is recognizable by some non-deterministic ω-ARNN*
2. $L \in \mathbf{\Sigma}^1_1$

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Conclusion

- Deterministic ω-ARNNs are strictly more expressive than any other usual deterministic abstract machines, in particular det Muller Turing machines.
- Non-deterministic ω-ARNNs are strictly more expressive than any other usual non-deterministic abstract machines, in particular non-det Muller Turing machines.