Limit Knowledge

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Agenda

- Introduction
- Aumann Structures
- Limit Knowledge
- Limit Knowledge and Games
- Agreeing to Disagree with Limit Knowledge
- Conclusion

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Interactive Epistemology

- This work fits in the context of *Interactive Epistemology*.
- *Interactive Epistemology* deals with the modelling of interactive knowledge and belief of multiple agents.
- *Interactive Epistemology* is a rather young discipline founded by Aumann (1976) and first been applied to games by Aumann (1987), and Tan and Werlang (1988).

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Epistemic Game theory

- *Epistemic Game Theory* complements game theory with the consideration of:
  - an *epistemic model* allowing to capture the interactive knowledge and belief of multiple agents;
  - *choice functions* allowing to connect the interactive epistemology to the game.
- Objectives of game theory:
  - epistemic foundations for existing solution concepts;
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Aumann Structures

We consider the so-called *set-based approach* to interactive epistemology as introduced by Aumann (1976).

Definition 1 (Aumann Structure)

An *Aumann structure* is a tuple $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p)$, where:

- Ω is a set of possible worlds;
- I is a set of agents;
- each $\mathcal{I}_i$ is a partition of Ω representing the information of agent i;
- $p : \Omega \to [0, 1]$ is a prior probability function such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

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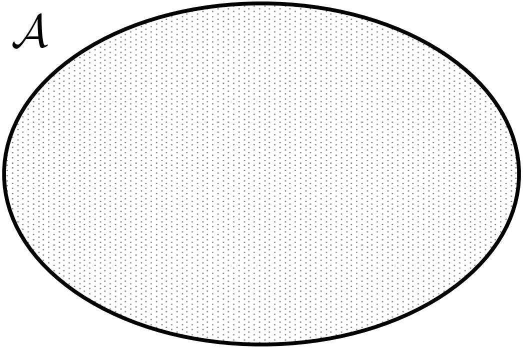

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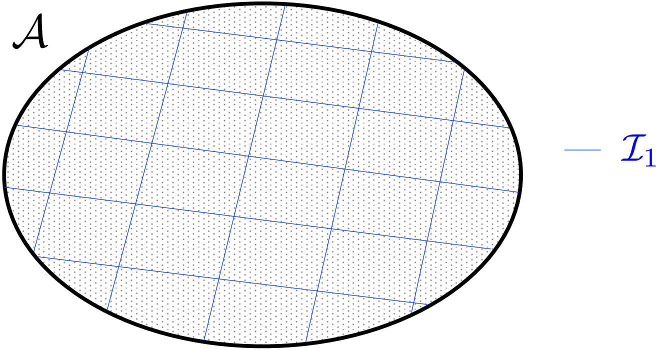

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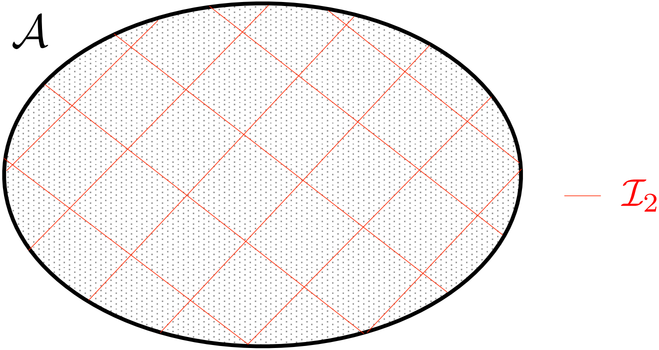

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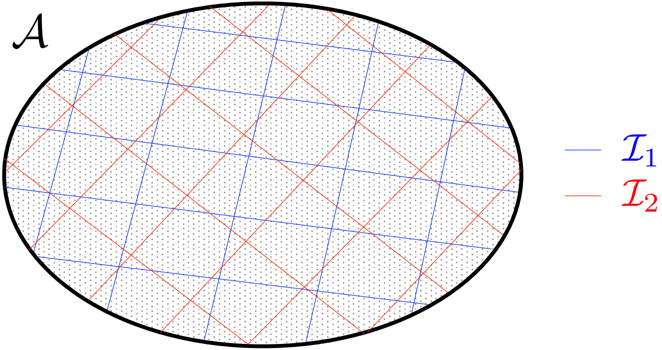
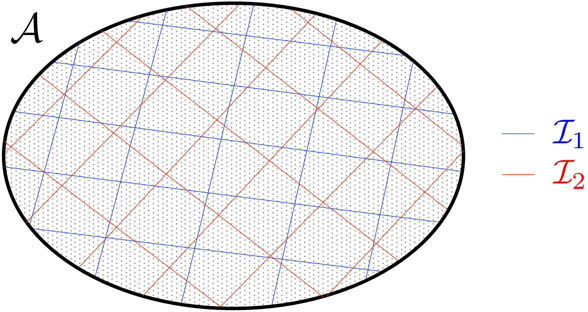


Events

In this framework, an *event* E is defined as a set of possible worlds of Ω.

The event "it is raining in London" consists of the set of all possible worlds where it does actually rain in London.



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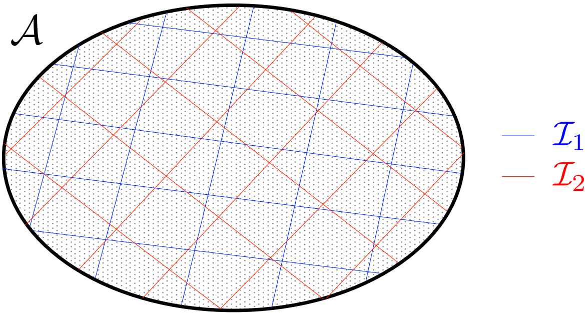

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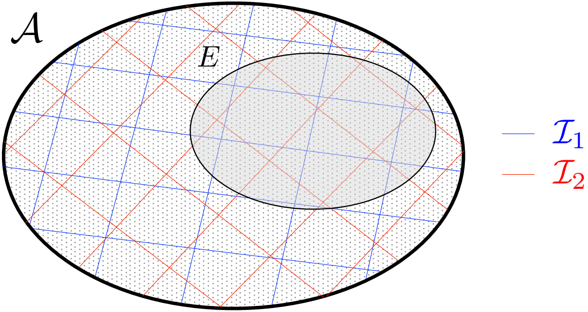
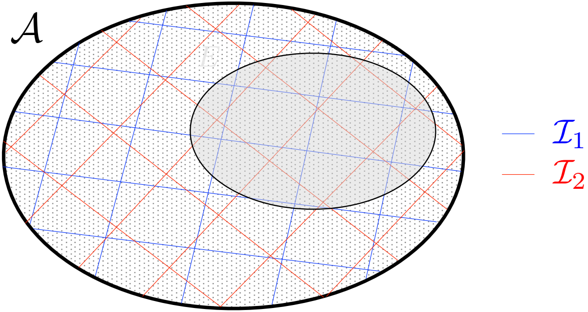


Knowledge

The event "Agent i knows E" is defined as

$$K_i(E) = \{\omega \in \Omega : \mathcal{I}_i(\omega) \subseteq E\}$$

Intuitively, agent i *knows* E iff in all worlds he considers possible, E holds.



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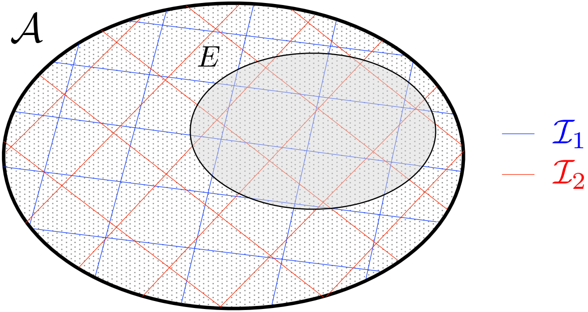

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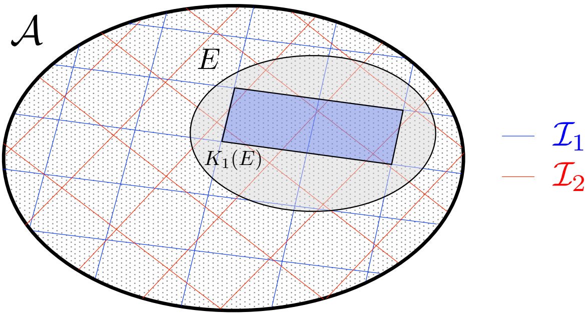

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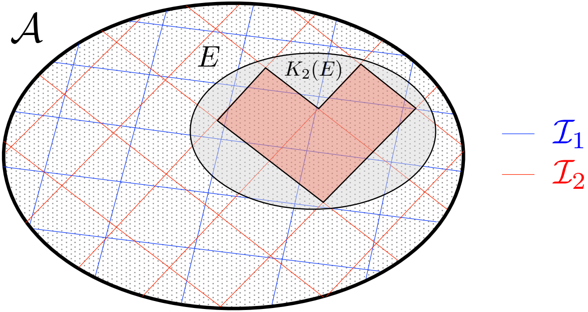

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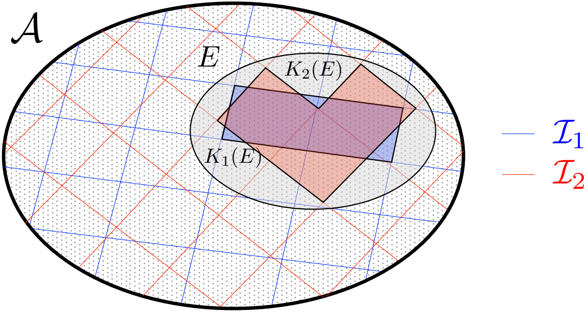
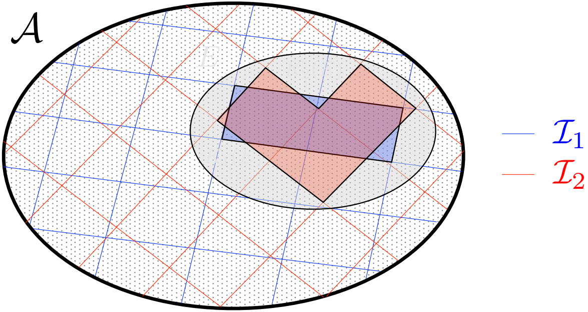


Mutual Knowledge

The *mutual knowledge* of E amongst the set I of agents is naturally defined by

$$K(E) = \bigcap_{i \in I} K_i(E)$$

The sequence of higher-order mutual knowledge of E is defined as $K^0(E) = E$ and $K^{m+1}(E) = K(K^m(E))$ for all $m > 0$.



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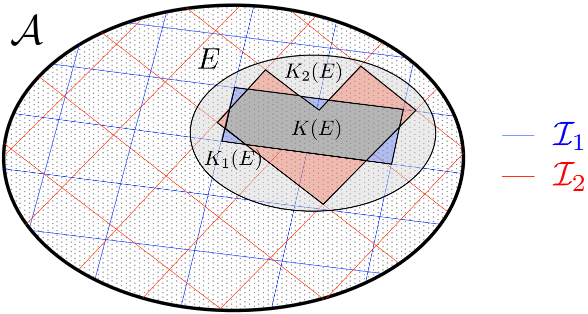

Common Knowledge

Common knowledge off E is defined as

$$CK(E) = \bigcap_{m \in \mathbb{N}} K^m(E).$$

On has the following properties:

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Beliefs

The *prior belief* function $p : \Omega \to [0,1]$ can naturally be extended to a *common prior belief* measure on the event space (also denoted p) $p : \mathcal{P}(\Omega) \to [0,1]$ defined by

$$p(E) = \Sigma_{\omega \in E} p(\omega), \text{ for any } E \subseteq \mathcal{P}(\Omega).$$

Moreover, all agents are assumed to be *Bayesian*. Hence, the *posterior belief* of agent i in event E at world ω is given by

$$p(E|\mathcal{I}_i(\omega)) = \frac{p(E \cap \mathcal{I}_i(\omega))}{p(\mathcal{I}_i(\omega))}.$$

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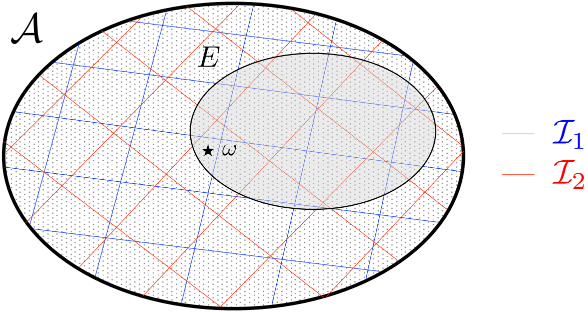

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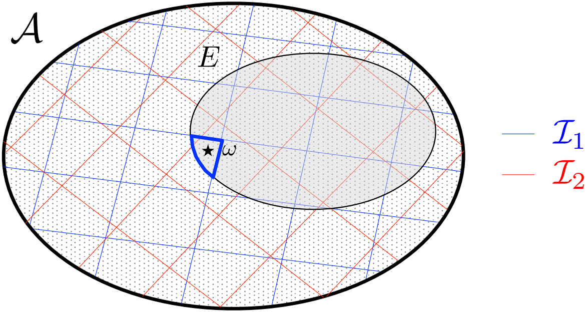

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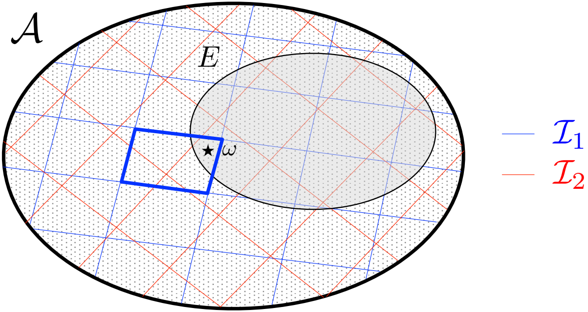

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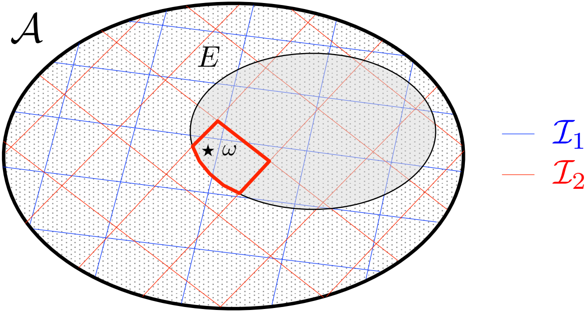

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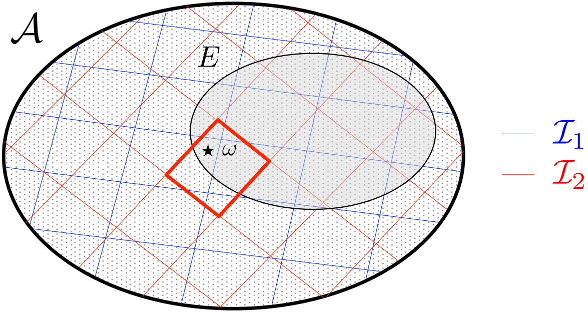

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The standard set-based approach to interactive epistemology lacks a general framework providing some formal notion of closeness between events.

An amended topological dimension introduces a perception of closeness between events permitting agents to reason deeper about knowledge and belief of events.

In such an enriched epistemic-topological framework, the reasoning of agents may depend on topological instead of mere epistemic features of the underlying interactive situation.

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Topological Aumann Structures

In this context, we consider the notion of a topological Aumann structure.

Definition 2 (Topological Aumann Structure)

A *topological Aumann structure* is a tuple $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p, \mathcal{T})$, where:

- $(\Omega, (\mathcal{I}_i)_{i \in I}, p)$ is a standard Aumann structure;
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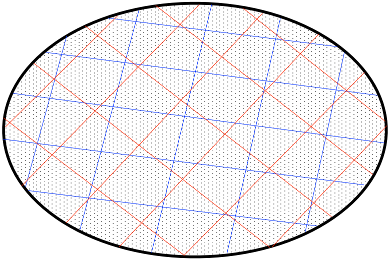

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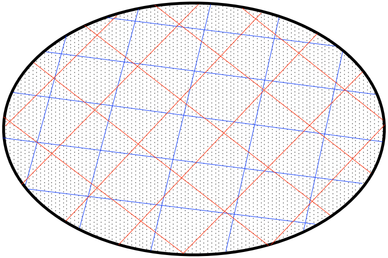
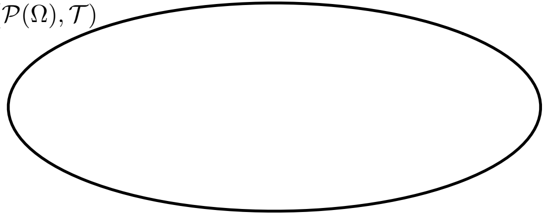


$(\mathcal{P}(\Omega), \mathcal{T})$



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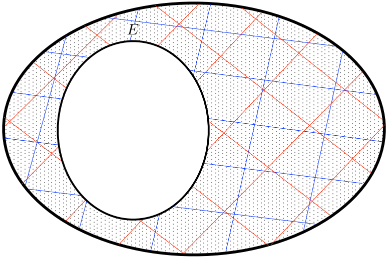

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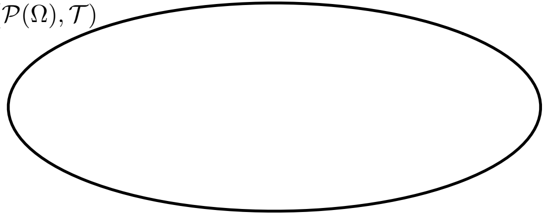

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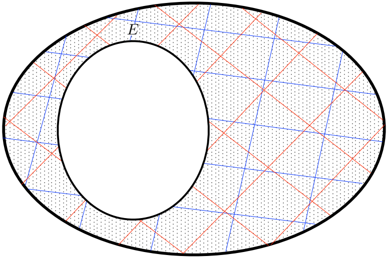

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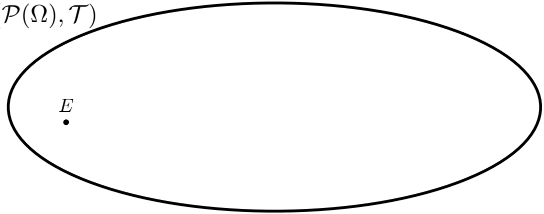

Limit Knowledge

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Definition 3 (Limit Knowledge)

Let $\mathcal{A}$ be a topological Aumann structure, and E be some event. If the (topological) limit point of the sequence of iterated mutual knowledge claims $(K^m(E))_{m>0}$ is unique, then

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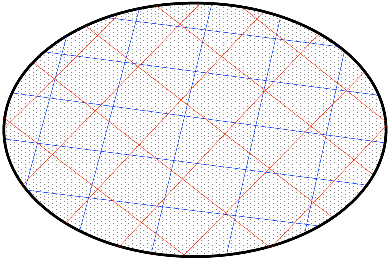

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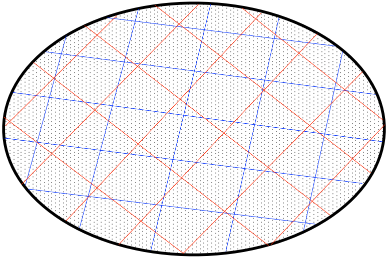

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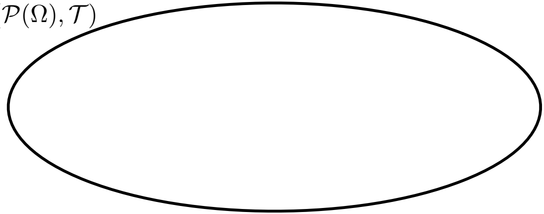

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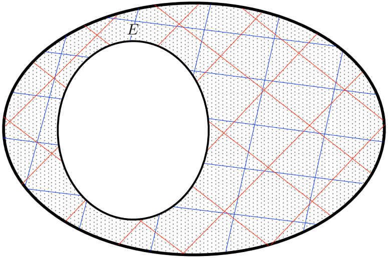

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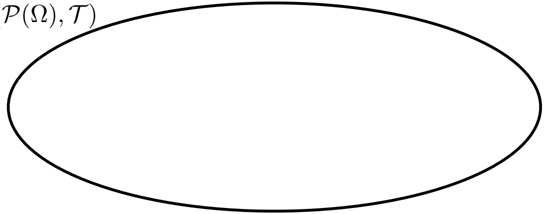

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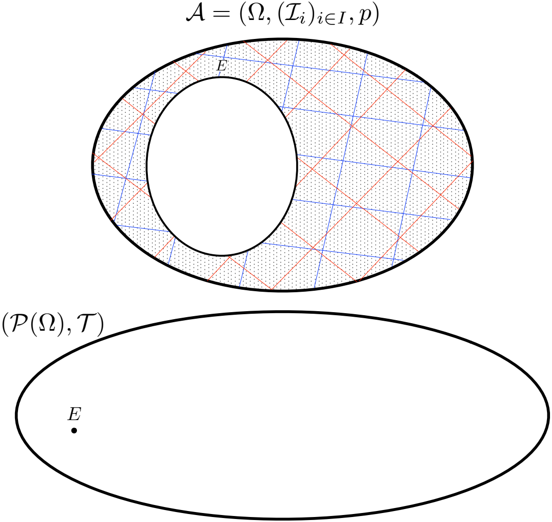

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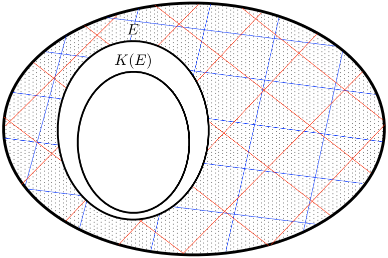

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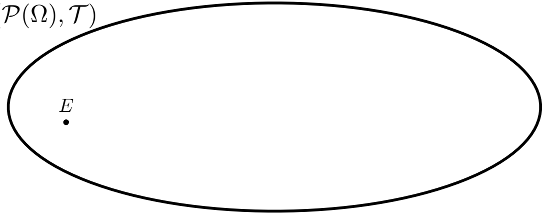

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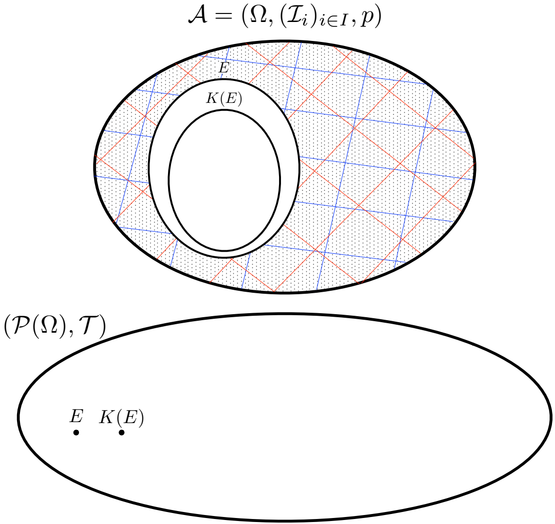

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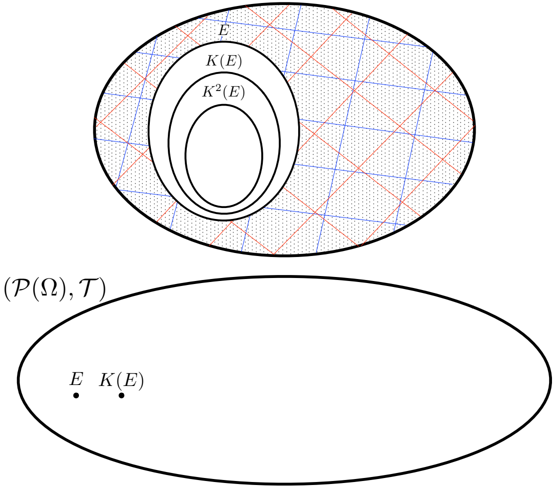

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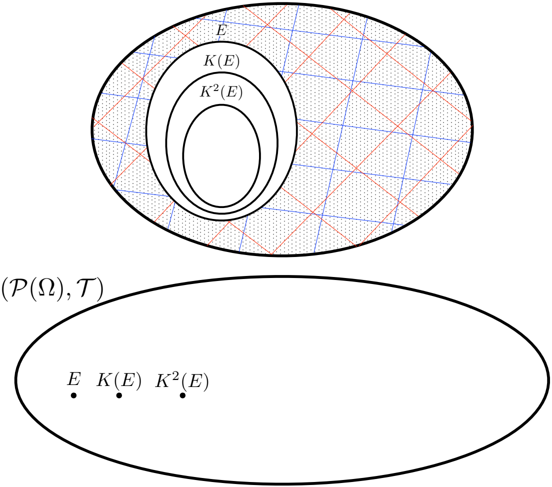

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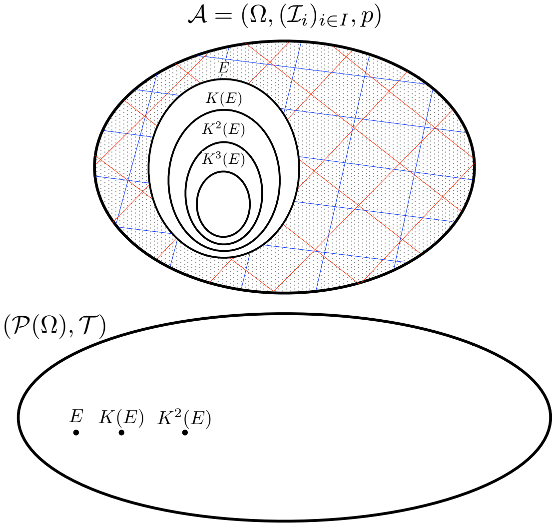

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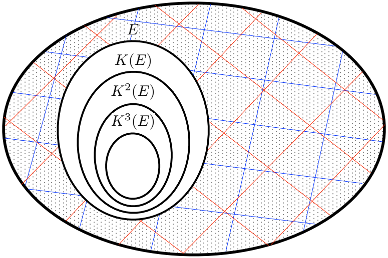

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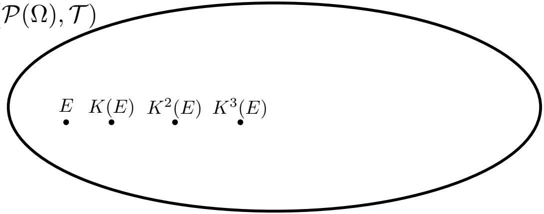

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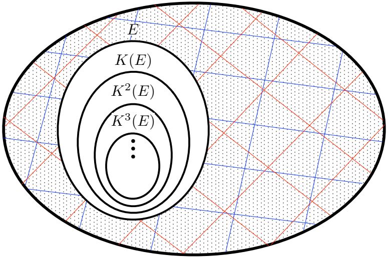

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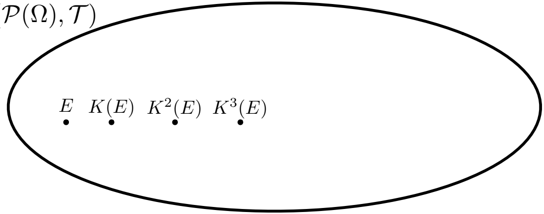

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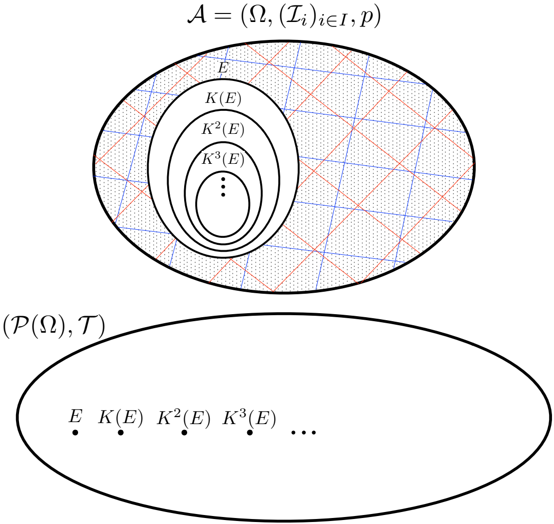

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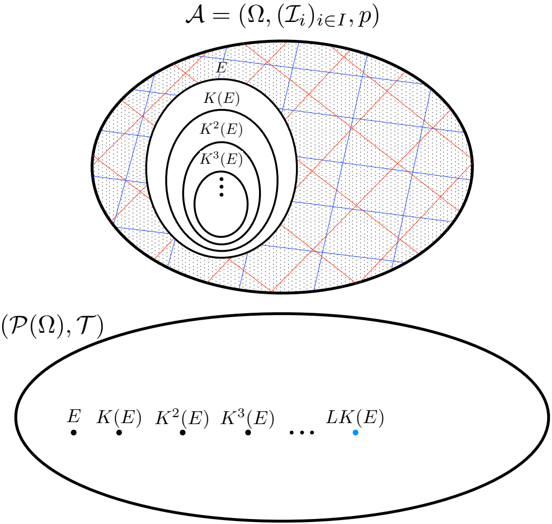

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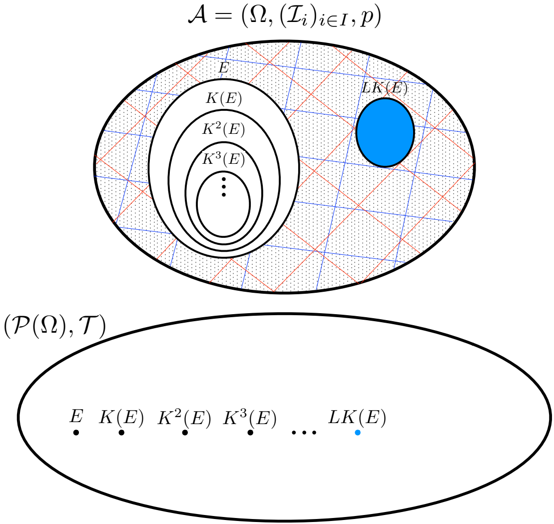

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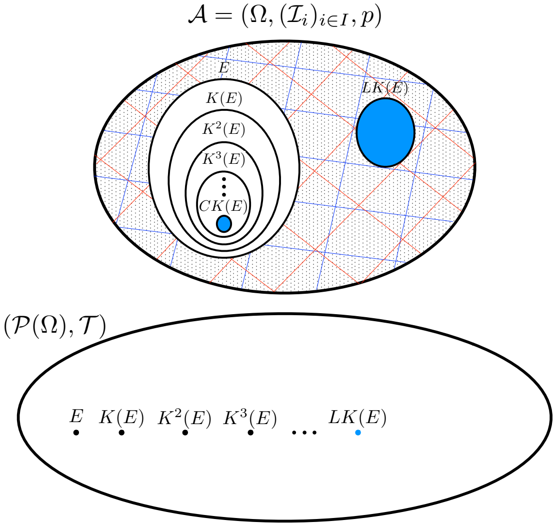

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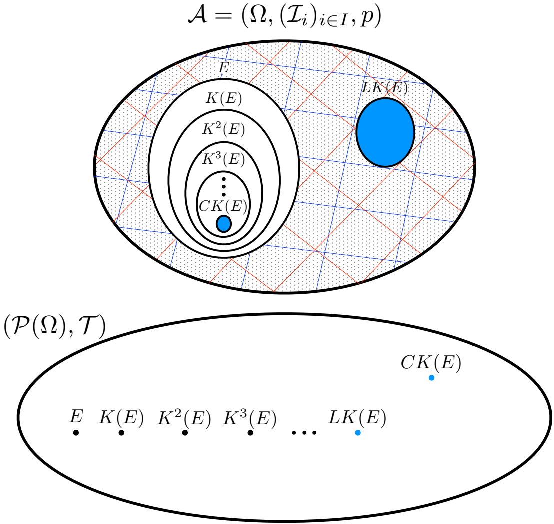

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- Limit knowledge of an event E is constituted by – whenever unique – the limit point of the sequence of iterated mutual knowledge, and thus linked to both epistemic as well as topological aspects of the event space.
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- Note that limit knowledge should not be amalgamated with common knowledge. Indeed, while common knowledge bears a standard implicative relation (in terms of set inclusion) to highest iterated mutual knowledge, limit knowledge entertains an implicative relation in terms of set proximity with highest iterated mutual knowledge.

Limit Knowledge vs Common Knowledge

The concept of limit knowledge clearly differs from that of common knowledge, but...

Lemma 4

Let $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p, \mathcal{T})$ be a topological Aumann structure and E be an event. If the sequence $(K^i(E))_{i \geq 0}$ of iterated mutual knowledge claims of E is eventually constant, then $CK(E)$ is a limit point of it.

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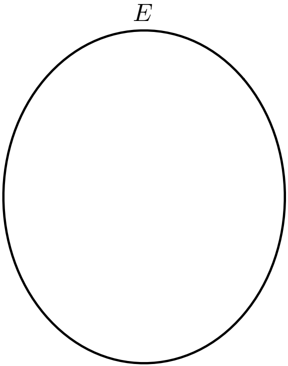

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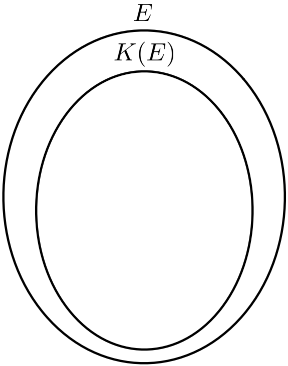

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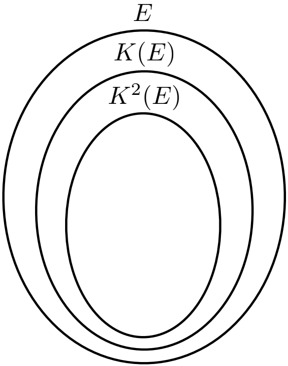

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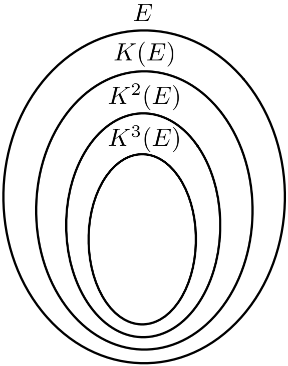

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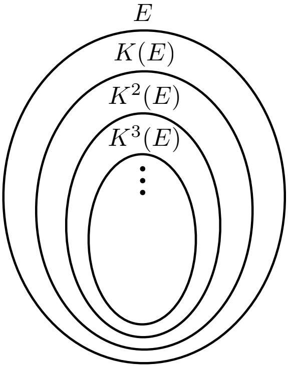

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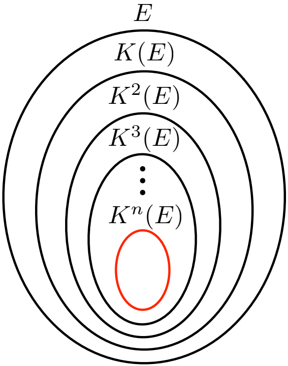

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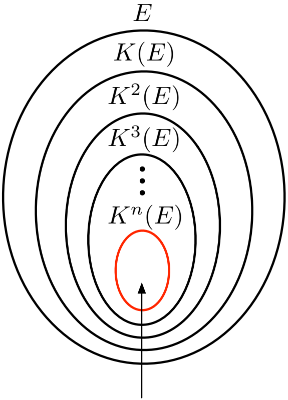

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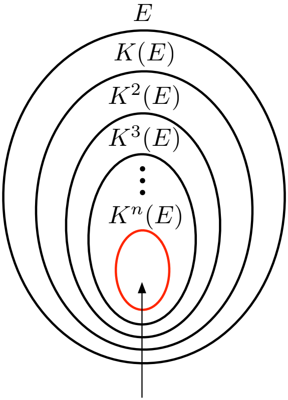

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Limit Knowledge vs Common Knowledge

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Proof of Corollary 5: Suppose that $\mathcal{A}$ is finite, i.e. that Ω is finite. Then $\mathcal{P}(\Omega)$ is also finite. Hence, the sequence $(K^i(E))_{i\geq 0}$ is eventually constant. By Lemma 4, $CK(E)$ is a limit point of $(K^i(E))_{i\geq 0}$.
In particular, if the sequence $(K^i(E))_{i\geq 0}$ has a unique limit point, namely $LK(E)$ by definition, then $CK(E) = LK(E)$. $\square$

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Lemma 6

Let $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I}, p, \mathcal{T})$ be a topological Aumann structure, where $\mathcal{T}$ is the discrete topology, and let E be an event. If $LK(E)$ is well-defined (i.e. $(K^i(E))_{i \geq 0}$ has a unique limit point), then $LK(E) = CK(E)$.

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Proof of Lemma 6: Suppose that $LK(E)$ is well-defined, i.e., is the unique limit point of $(K^i(E))_{i\geq 0}$. Hence, for any neighbourhood N of $LK(E)$, there exists an index p_N such that all elements of the sequence belong to N from index p_N onwards. By definition of the discrete topology $\mathcal{T}$, every subset of $\mathcal{P}(\Omega)$ is open. In particular, the singleton $N = \{LK(E)\}$ is an open neighbourhood of $LK(E)$. Hence, there exists p_N such that $K^i(E) \in N$ for all $i \geq p_N$. This means $K^i(E) = LK(E)$ for all $i \geq p_N$. Therefore $CK(E) := \bigcap_{i\geq p_N} K^i(E) = LK(E)$. $\square$

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The concept of limit knowledge clearly differs from that of common knowledge, but... in order for limit knowledge to be distinct from common knowledge (hence possibly interesting), the following conditions need to be satisfied:

- the underlying topological Aumann structure needs to be *infinite* (Corollary 5);
- the sequence of iterated mutual knowledge claims needs to be *strictly shrinking* (Lemma 4);
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Limit Knowledge and Games

The operator limit knowledge is capable of providing alternative epistemic-topological characterizations of solution concepts in games.

We give an example of a game and an epistemic-topological model of it where *limit knowledge of rationality* is a strict refinement of *common knowledge of rationality* in terms of solution concepts.

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Game

We first recall some basic definitions...

Definition 7 (Game)

A *game in normal form* is a tuple $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ where:

- I is a set of players;
- each S_i is a strategy space for player i;
- each $u_i : \times_{i \in I} S_i \to \mathbb{R}$ a utility function that assigns for each strategy profile $(s_i)_{i \in I} \in \times_{i \in I} S_i$ a real number $u_i((s_i)_{i \in I})$ as payoff.

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Solution Concept

Definition 8 (Solution Concept)

A *solution concept* $\mathcal{SC}$ is a mapping associating with each game Γ a subset of its strategy profiles $\mathcal{SC}^{\Gamma} \subseteq \times_{i \in I} S_i$.

Note that a solution concept is a generic method which does not depend on any particular given game.

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Iterated Strict Dominance

Definition 9 (Iterated Strict Dominance)

Let $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ be a game. Let the sequence $(SD_i^k)_{k \geq 0}$ be inductively defined for every player $i \in \mathcal{I}$ and $k \geq 0$ by

- $SD_i^0 := S_i$
- $SD_i^{k+1} := SD_i^k \setminus \{s_i \in SD_i^k : \exists s_i' \in SD_i^k \, \forall s_{-i} \in SD_{-i}^k \text{ s.t. } u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})\}$

Let further $SD^k = \times_{i \in \mathcal{I}} SD_i^k$. The solution concept *iterated strict dominance* is then given by $ISD^\Gamma := \bigcap_{k \geq 0} SD^k$.

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Epistemic Model of a Game

Definition 10 (Epistemic Model)

An *epistemic model* of a game Γ is an Aumann structure $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i \in I}, (\sigma_i)_{i \in I})$ that additionally specifies for each player $i \in I$ a *choice function* $\sigma_i : \Omega \to S_i$, connecting the interactive epistemology to the game.

The *choice function* profile $\sigma : \Omega \to \times_{i \in I} S_i$ mapping each world to its corresponding strategy profile is then defined by $\sigma(\omega) = (\sigma_i(\omega))_{i \in I}$.

Moreover, it is standard and natural to assume that each player knows his own strategy choice (*measurability assumption*), i.e., if two worlds ω and ω' are such that $\mathcal{I}_i(\omega) = \mathcal{I}_i(\omega')$, then $\sigma_i(\omega) = \sigma_i(\omega')$.

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Knowledge-Based Rationality

We consider the following notion of knowledge-based rationality:

Definition 11 (Knowledge-Based Rationality)

Let Γ be a game and $\mathcal{A}^\Gamma$ be an epistemic model of it. The event *player i is rational* is defined as

$$R_i := \bigcap_{s_i \in S_i} \left(\Omega \setminus K_i\{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega) > u_i(\sigma(\omega))\} \right).$$

The event that *all players are rational* is called *rationality* and defined as $R := \bigcap_{i \in I} R_i$.

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$CK(R)$ implies ISD

Common knowledge of knowledge-based rationality implies iterated strict dominance.

Proposition 12

Let Γ be a game and $\mathcal{A}^\Gamma$ be an epistemic model of it. Then $\sigma(CK(R)) \subseteq ISD^\Gamma$.

Proof: By induction, we show that $\sigma(K^m(R)) \subseteq SD^{m+1}$, for all $m \geq 0$. It follows that $\sigma(CK(R)) = \sigma(\bigcap_{m\geq 0} K^m(R)) \subseteq \bigcap_{m\geq 0} \sigma(K^m(R)) \subseteq \bigcap_{m\geq 0} SD^{m+1}(R) = ISD^\Gamma$. $\square$

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Proposition 12

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Proof: By induction, we show that $\sigma(K^m(R)) \subseteq SD^{m+1}$, for all $m \geq 0$. It follows that $\sigma(CK(R)) = \sigma(\bigcap_{m\geq 0} K^m(R)) \subseteq \bigcap_{m\geq 0} \sigma(K^m(R)) \subseteq \bigcap_{m\geq 0} SD^{m+1}(R) = ISD^\Gamma$. $\square$

Example of a Cournot-type game

We provide a game Γ with an epistemic model of it $\mathcal{A}^\Gamma$ such that:

- iterated dominance followed by weak dominance is a strict refinement of iterated strict dominance, i.e.,

$$(ISD + WD)^\Gamma \subsetneq ISD^\Gamma$$

- common knowledge of rationality reveals precisely the strategies that survive iterated strict dominance, i.e.,

$$\sigma(CK(R)) = ISD^\Gamma$$

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Example of a Cournot-type game

Consider the Cournot-type game $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ where

- $I = \{Alice, Bob, Claire, Donald\}$
- $S_{Alice} = S_{Bob} = [0, 1]$, $S_{Claire} = \{U, D\}$, $S_{Donald} = \{L, R\}$
- $u_i : S_{Alice} \times S_{Bob} \times S_{Claire} \times S_{Donald} \to \mathbb{R}$ for all $i \in I$, and

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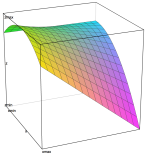
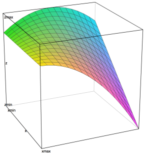
- $u_i : S_{Alice} \times S_{Bob} \times S_{Claire} \times S_{Donald} \to \mathbb{R}$, for all $i \in I$, are:

$u_{Alice}(x,y,v,w) = x(1-x-y)$ $\quad$ $u_{Bob}(x,y,v,w) = y(1-x-y)$





$u_{Claire}(x,y,v,w)$ and $u_{Donald}(x,y,v,w)$ are given by:

		Donald: L	Donald: R
Claire	U	$(2,1)$	$(1,1)$
	D	$(2,2)$	$(2,3)$

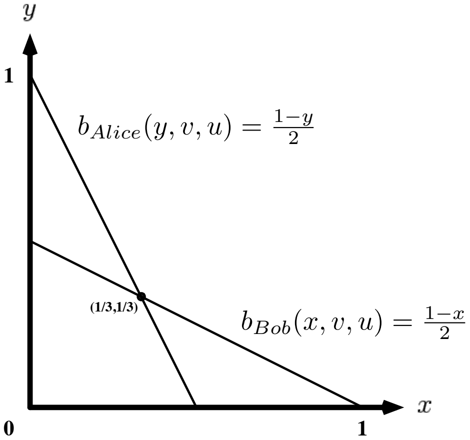
for all $(x,y) \neq (\frac{1}{3}, \frac{1}{3})$

		Donald: L	Donald: R
Claire	U	$(2,3)$	$(2,2)$
	D	$(1,1)$	$(2,1)$

for $(x,y) = (\frac{1}{3}, \frac{1}{3})$

Example of a Cournot-type game

The best-response functions of *Alice* and *Bob* are given by:



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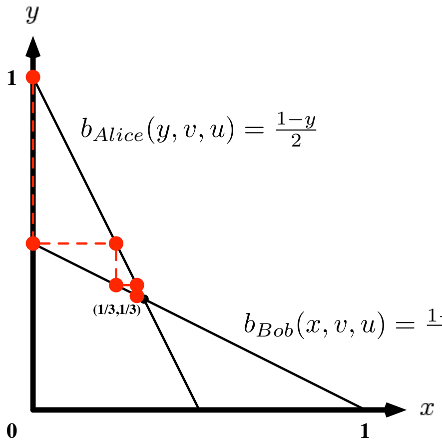

Example of a Cournot-type game

For this game, one has:

- $ISD^{\Gamma} = \{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\}$
- $(ISD + WD)^{\Gamma} = \{\frac{1}{3}, \frac{1}{3}, U, L\}$

Hence, $(ISD + WD)^{\Gamma}$ provides a strict refined strategy profile set than that induced by ISD^{Γ}.

Donald

Claire		L	R
	U	$(2, 3)$	$(2, 2)$
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for $(x, y) = (\frac{1}{3}, \frac{1}{3})$

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for $(x, y) = (\frac{1}{3}, \frac{1}{3})$

Example of a Cournot-type game

We consider the epistemic model $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i \in I}, (\sigma_i)_{i \in I})$ given by (c.f. next slide):

- $\Omega = \{\alpha, \beta, \gamma, \delta\} \cup \{\alpha_i, \beta_i, \gamma_i, \delta_i : i \geq 0\}$
- $\mathcal{I} = \{Alice, Bob, Claire, Donald\}$
- $(\mathcal{I}_i)_{i \in I}$ are as described in next slide...
- $\sigma = (\sigma_{Alice}, \sigma_{Bob}, \sigma_{Claire}, \sigma_{Donald}) : \Omega \to \times_{i \in I}$ is given by:

$$\begin{array}{ll}
\sigma(\alpha) = (1/3, 1/3, U, L) & \sigma(\alpha_n) = (s^n_{Alice}, s^n_{Bob}, U, L) \\
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Example of a Cournot-type game

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Aumann structure

α	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	$\cdots$
β	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	$\cdots$
γ	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	$\cdots$
δ	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	$\cdots$

Example of a Cournot-type game

Aumann structure — *Alice*

$$\boxed{\begin{array}{c}\alpha\\ \beta\\ \gamma\\ \delta\end{array}} \quad \boxed{\begin{array}{cc}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\\ \gamma_0 & \gamma_1\\ \delta_0 & \delta_1\end{array}} \quad \boxed{\begin{array}{cc}\alpha_2 & \alpha_3\\ \beta_2 & \beta_3\\ \gamma_2 & \gamma_3\\ \delta_2 & \delta_3\end{array}} \quad \boxed{\begin{array}{cc}\alpha_4 & \alpha_5\\ \beta_4 & \beta_5\\ \gamma_4 & \gamma_5\\ \delta_4 & \delta_5\end{array}} \quad \boxed{\begin{array}{cc}\alpha_6 & \alpha_7\\ \beta_6 & \beta_7\\ \gamma_6 & \gamma_7\\ \delta_6 & \delta_7\end{array}} \quad \begin{array}{c}\cdots\\ \cdots\\ \cdots\\ \cdots\end{array}$$

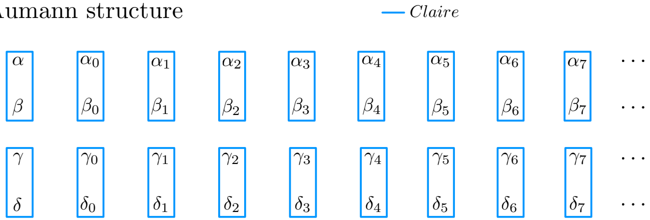
Example of a Cournot-type game

Aumann structure — *Bob*

α	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	$\cdots$
β	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	$\cdots$
γ	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	$\cdots$
δ	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	$\cdots$

Example of a Cournot-type game

Aumann structure



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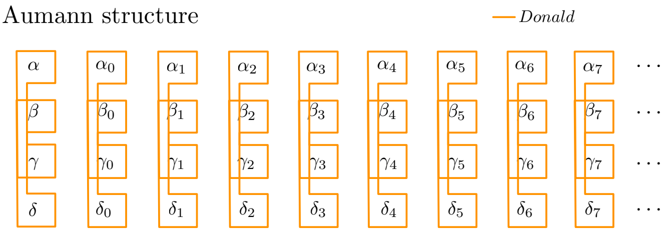

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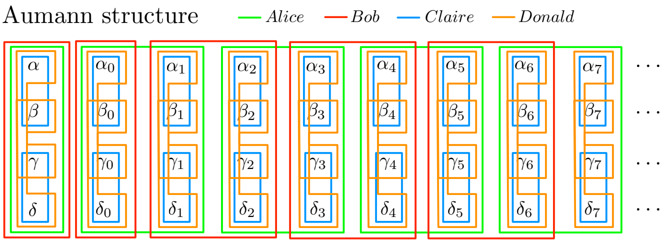

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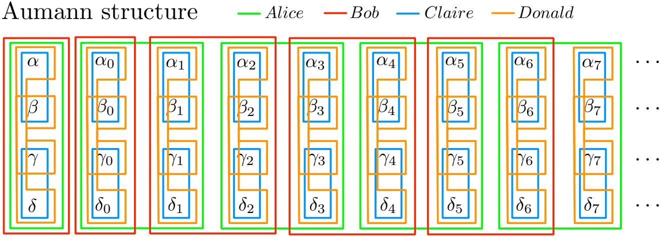
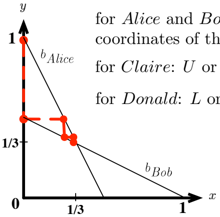


Strategies



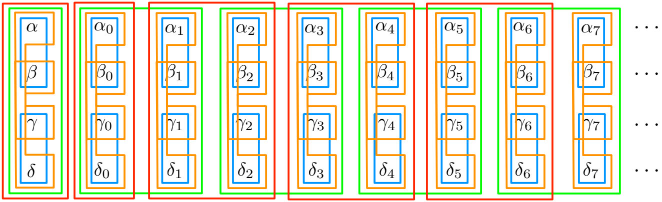
for *Alice* and *Bob*:
coordinates of the red points...

for *Claire*: U or D

for *Donald*: L or R

Example of a Cournot-type game

Aumann structure — *Alice* — *Bob* — *Claire* — *Donald*



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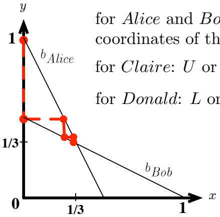

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Choice function

$\sigma(\alpha) = (1/3, 1/3, U, L)$
$\sigma(\beta) = (1/3, 1/3, U, R)$
$\sigma(\gamma) = (1/3, 1/3, D, L)$
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Example of a Cournot-type game

For this game, one has $R = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$, hence $CK(R) = \{\alpha, \beta, \gamma, \delta\}$.

Moreover, suppose $\mathcal{P}(\Omega)$ equipped with the topology

$$\mathcal{T} = \{O \subseteq \mathcal{P}(\Omega) : \{\alpha\} \notin O\} \cup \{\mathcal{P}(\Omega)\}$$

It follows that $LK(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = \{\alpha\}$. Hence,

$$\begin{aligned}
\sigma(CK(R)) &= \{\sigma(\alpha), \sigma(\beta), \sigma(\gamma), \sigma(\delta)\} \\
&= \{1/3\} \times \{1/3\} \times \{U, D\} \times \{L, R\} = ISD^{\Gamma} \\
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Therefore, the solution in accordance with $LK(R)$ is a strict refinement of the solution induced by $CK(R)$.

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Limit Knowledge and Games

Moreover, *limit knowledge of rationality* is potentially capable of characterizing any possible event and solution concept.

Theorem 13

Let Γ be a normal form game and A^Γ an epistemic model of it such that $(K^m(R))_{m>0}$ is strictly shrinking (where R is the event "rationality").

1. *Let E be any event. Then, there exists a topology on $P(\Omega)$ such that $LK(R) = E$.*
2. *Let SC be any solution concept. Then, there exists a topology on $P(\Omega)$ such that $\sigma(LK(R)) \subseteq SC^\Gamma$.*

Limit Knowledge and Games

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Proof of Theorem 13. (Point 2.) Let $F = \sigma^{-1}(\mathcal{SC}^\Gamma)$. Then $\sigma(F) \subseteq \mathcal{SC}^\Gamma$. Suppose the event space $\mathcal{P}(\Omega)$ equipped with the excluded-point topology

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Towards the issue of *plausible* topological considerations, we can for instance consider the following topologies:

- Partition topology (on the state space): reflects informational indistinguishability between possible worlds in terms of separation properties.
- Common truism topology (on the state space): reflects high-indistinguishability between possible worlds in terms of separation properties.
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Agreeing to disagree with Limit Knwoledge

Limit Knowldege enables to revisit Aumann's famous "no agreeing to disagree theorem" from an epistemic-topological perspective.

More precisely, if the original hypotheses of Aumann's theorem are modified in that the epistemic operator *common knowledge* is replaced by the epistemic-topological operator *limit knowledge*, then agents can indeed agree to disagree.

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Aumann's agreement theorem states that if two agents entertain a common prior belief function and their posterior beliefs in some event are common knowledge, then these posterior beliefs must coincide.

In other words, if two agents with common prior beliefs hold distinct posterior beliefs, then these posteriors cannot be common knowledge among them.

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Aumann's Theorem can be formalised as follows:

Theorem 14 (Aumann (1976))

Let $\mathcal{A}$ be an Aumann structure, $E \subseteq \Omega$ be an event and $\hat{\omega} \in \Omega$ be a world such that

$$CK\left(\bigcap_{i=1}^{n}\{\omega' \in \Omega : p(E \mid \mathcal{I}_i(\omega')) = p(E \mid \mathcal{I}_i(\hat{\omega}))\}\right) \neq \emptyset,$$

then $p(E \mid \mathcal{I}_1(\hat{\omega})) = p(E \mid \mathcal{I}_2(\hat{\omega})) = \cdots = p(E \mid \mathcal{I}_n(\hat{\omega}))$.

Moreover, if

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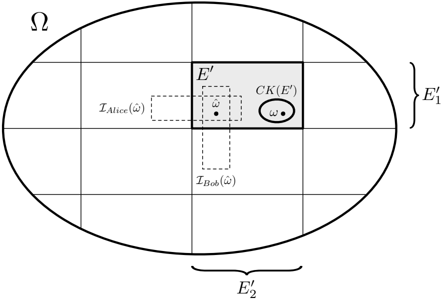

$$E'_1 = \{\omega' \in \Omega : p(E \mid \mathcal{I}_{Alice}(\omega')) = p(E \mid \mathcal{I}_{Alice}(\hat{\omega}))\}$$

$$E'_2 = \{\omega' \in \Omega : p(E \mid \mathcal{I}_{Bob}(\omega')) = p(E \mid \mathcal{I}_{Bob}(\hat{\omega}))\}$$

$$E' = E'_1 \cap E'_2$$

If $CK(E')$ is non-empty, then the posteriors beliefs in E of $Alice$ and Bob at $\hat{\omega}$ cannot differ. In fact, the posteriors of $Alice$ and Bob also coincide at ω.

Aumann's Theorem

- Along the lines of Aumann's theorem, Milgrom and Stokey (1982) establish an impossibility theorem of speculative trade. Intuitively, their result states that if two traders agree on a prior efficient allocation of goods, then upon receiving private information it cannot be common knowledge that both traders have an incentive to trade.
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We revisited Aumann's theorem from the "limit knowledge" perspective, and proved that it is possible for agent to "limit-agree to disagree".

More precisely, if the original hypotheses of Aumann's result are modified in that the epistemic operator common knowledge is replaced by the epistemic-topological operator limit knowledge, then agents can indeed agree to disagree.

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There exist an Aumann structure $\mathcal{A}$ equipped with a topology $\mathcal{T}$ on the event space $\mathcal{P}(\Omega)$, an event $E \subseteq \Omega$, and worlds $\omega, \hat{\omega} \in \Omega$ such that

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as well as $p(E \mid \mathcal{I}_i(\omega)) \neq p(E \mid \mathcal{I}_j(\omega))$ for some agents $i, j \in I$.

In other words, agents limit-agree on their posteriors in E at ω, and these posteriors are nevertheless distinct.

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$\boldsymbol{w_0}$ w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 $\dots$

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Aumann structure — *Alice*

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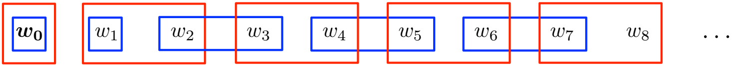

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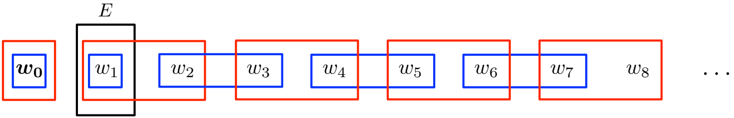

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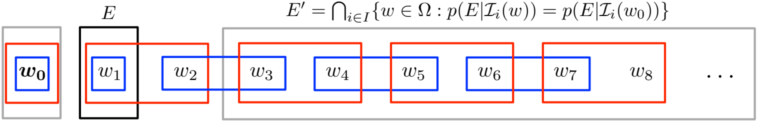

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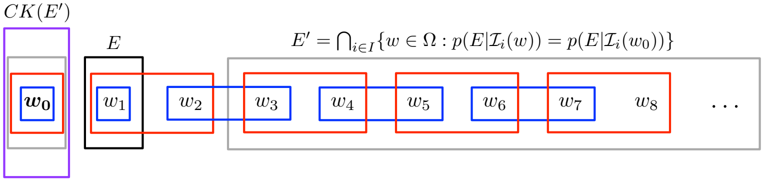

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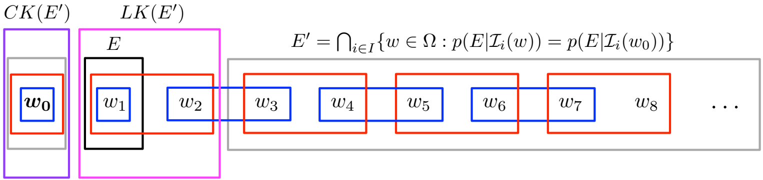

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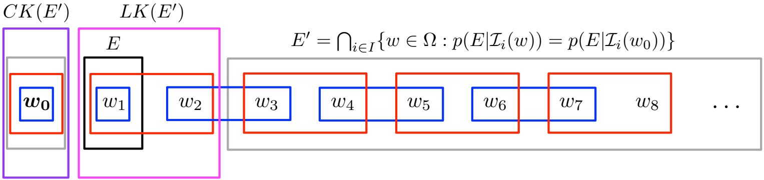

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One has, $w_1 \in LK(E')$ as well as $p(E|\mathcal{I}_{Alice}(w_1)) = 1 \neq \frac{2}{3} = p(E|\mathcal{I}_{Bob}(w_1))$.
Hence, at w_1, agents limit-agree on distinct posterior beliefs.
Agents limit-agree to disagree.

Conclusion

- Limit knowledge is a new epistemic-topological operator which captures reasoning patterns of agents base on closeness of events.
- The operator limit knowledge is capable of providing relevant epistemic-topological characterizations of solution concepts in games.
- With limit knowledge, a "limit-agreeing to disagree" theorem is possible.
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- The operator limit knowledge is capable of providing relevant epistemic-topological characterizations of solution concepts in games.
- With limit knowledge, a "limit-agreeing to disagree" theorem is possible.
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