Expressive Power of Recurrent Neural Networks over Infinite Input Streams

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Introduction

- The computational capabilities of recurrent neural networks have mainly been studied in the context of classical computation: McCulloch & Pitts, Turing, Kleene, von Neumann, Minsky, Papert,. . . , Siegelmann & Sontag,. . .
- We provide a characterization of the computational power of recurrent neural networks in terms of their attractor dynamics, i.e., in the context of infinite input stream computation.

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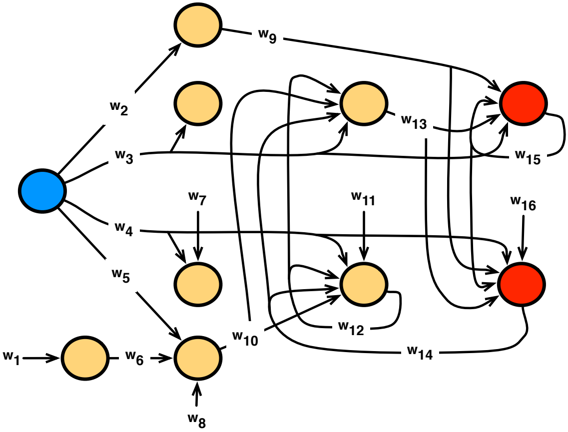
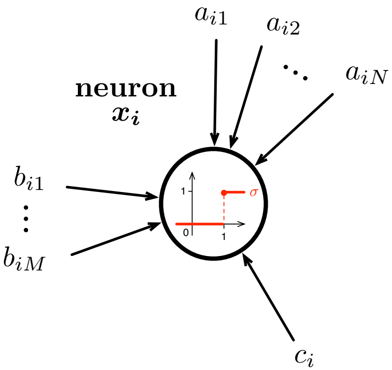
Recurrent Neural Network



Boolean Recurrent Neural Networks



$$x_i(t+1) = \theta\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

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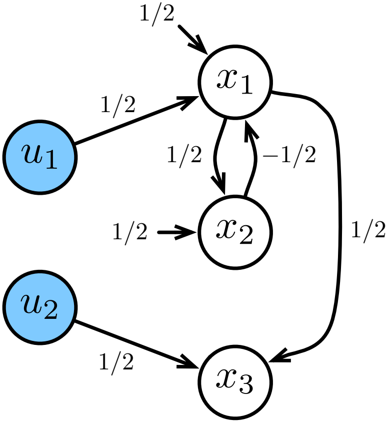

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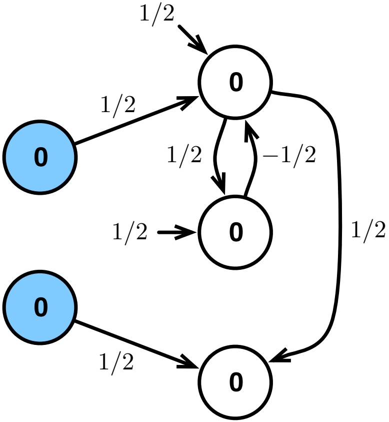

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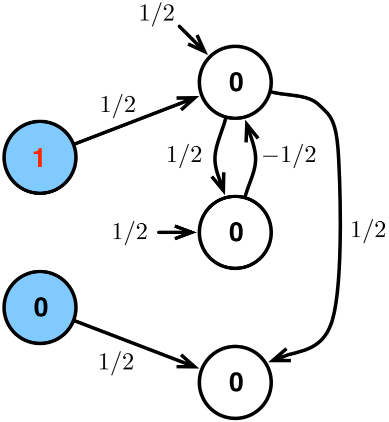

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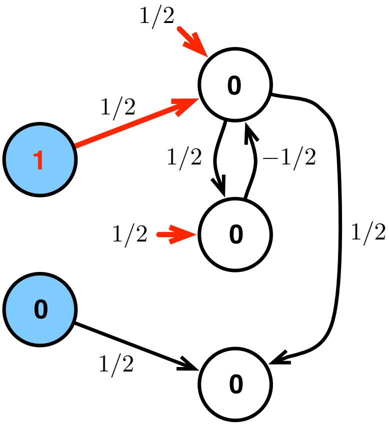

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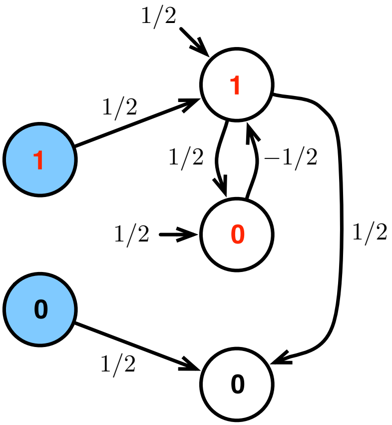

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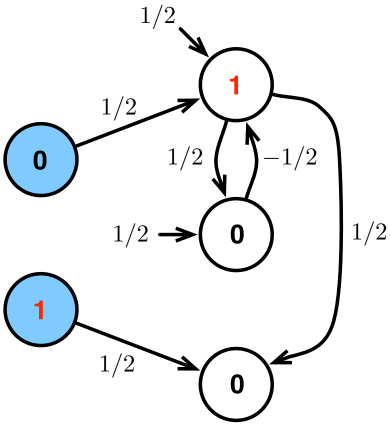
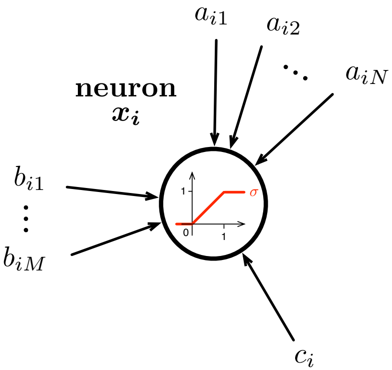


Sigmoidal Recurrent Neural Networks



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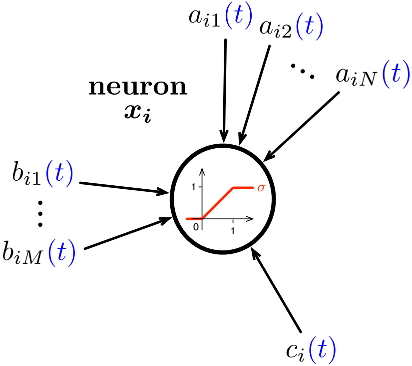

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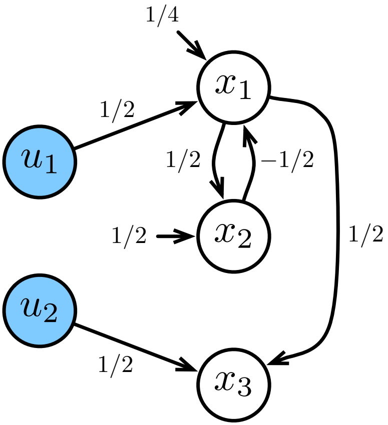

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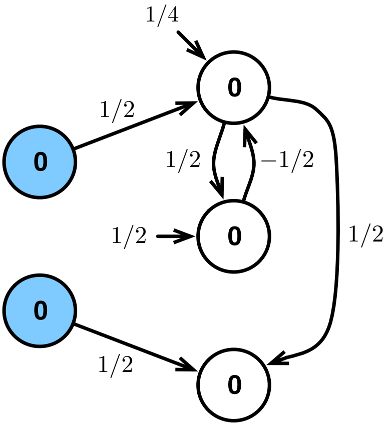

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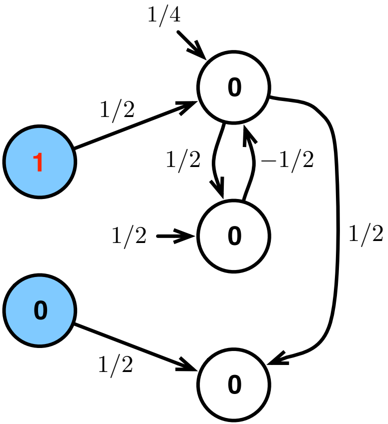

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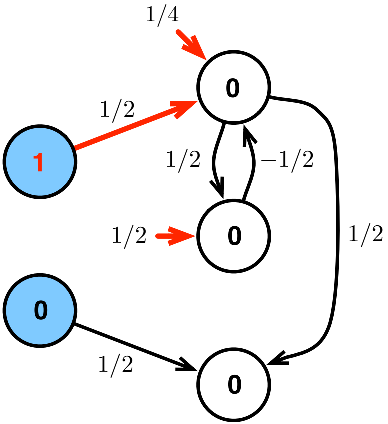

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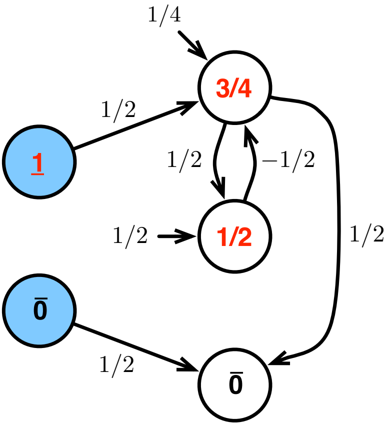

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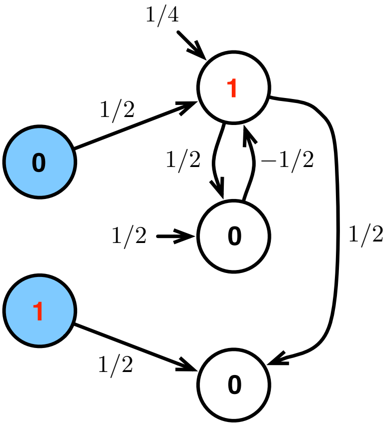

Recurrent Neural Networks

We consider eight models of RNNs:

1. Boolean rational RNNs: B-RNN[$\mathbb{Q}$]s
2. Boolean real RNNs: B-RNN[$\mathbb{R}$]s
3. Sigmoidal static rational RNNs: St-RNN[$\mathbb{Q}$]s
4. Sigmoidal static real RNNs: St-RNN[$\mathbb{R}$]s
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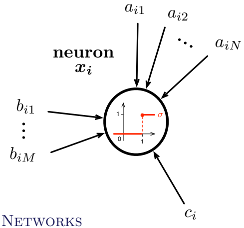

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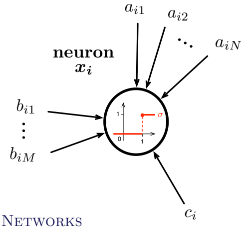

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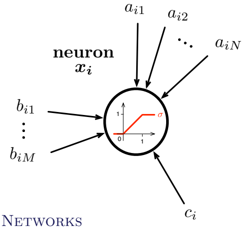

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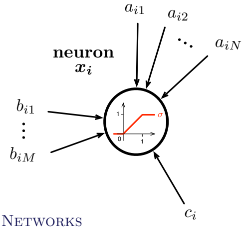

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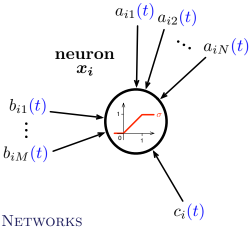

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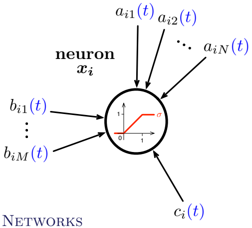

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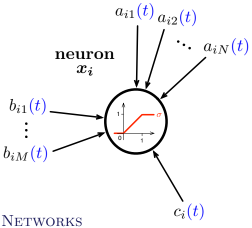

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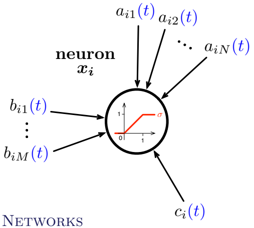

Results (Classical Computation)

	Boolean	Static	Bi-valued Evolving	Evolving
$\mathbb{Q}$	FSA	TM	TM/poly(A)	TM/poly(A)
	REG	**P**	**P/poly**	**P/poly**
	Kl 56, Mi 67	Si & So 95	Ca & Si 11,14	Ca & Si 11,14
$\mathbb{R}$	FSA	TM/poly(A)	TM/poly(A)	TM/poly(A)
	REG	**P/poly**	**P/poly**	**P/poly**
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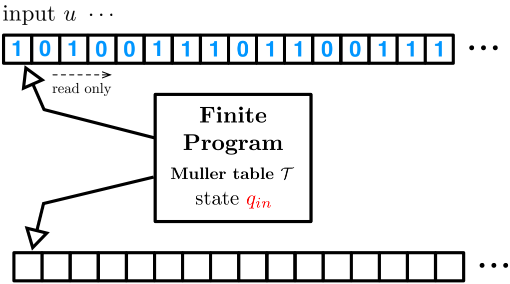
Muller Turing Machine

A *Muller Turing machine* consists of a classical TM with Muller acceptance condition. Muller table $\mathcal{T}$: collection of sets of states.



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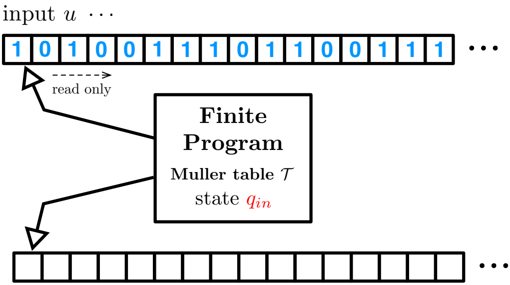

- The ω-word u is *accepted* by $\mathcal{M}$ if $\inf(\rho_u) \in \mathcal{T}$.

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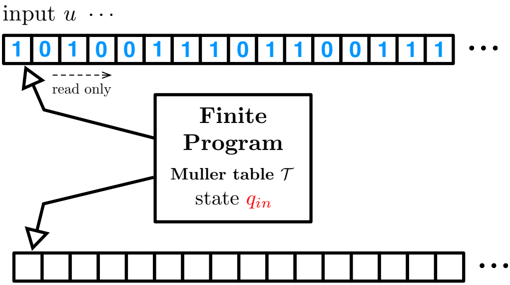

- The ω-word u is *rejected* by $\mathcal{M}$ if $\inf(\rho_u) \notin \mathcal{T}$.

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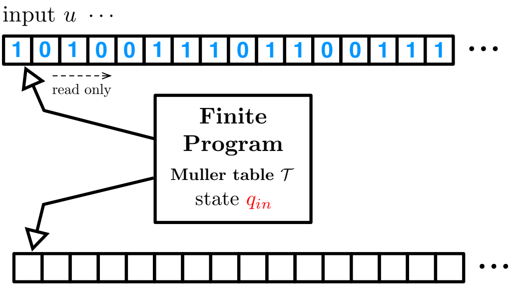
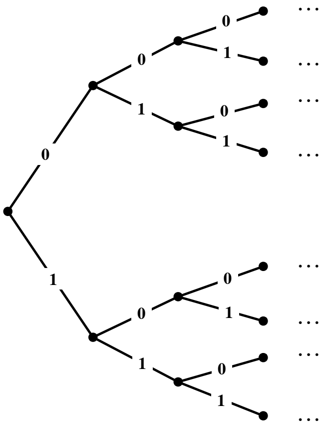


- Set of ω-words accepted is the ω-language *recognized* by $\mathcal{M}$.

Topology

The Cantor space $\{0,1\}^\omega$
the set of infinite sequences of bits



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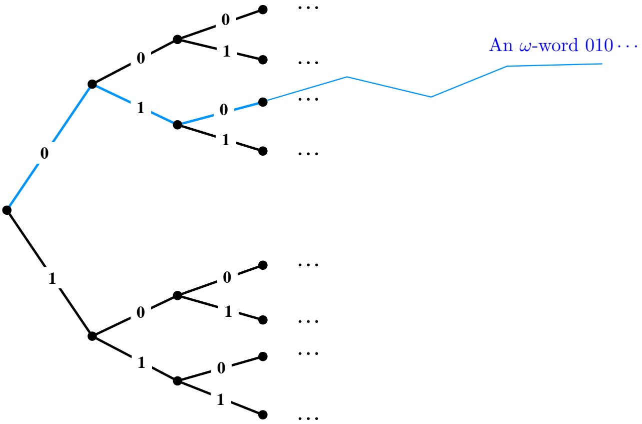

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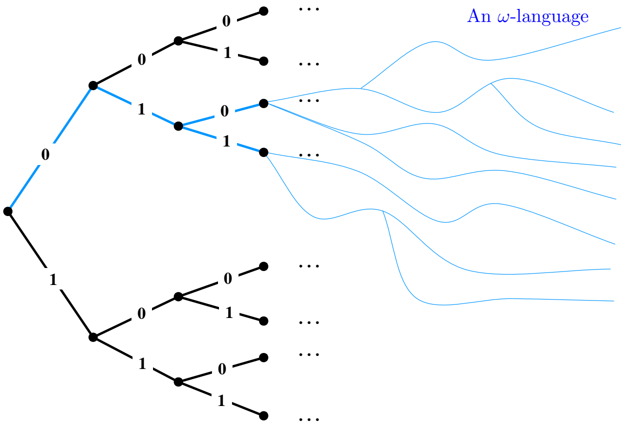

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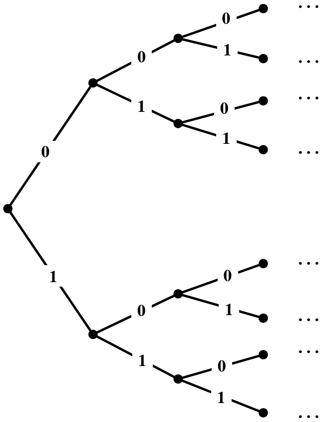
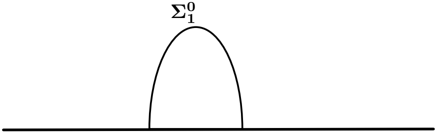


equipped with product topology
of the discrete topology on $\{0,1\}$

Topology

Topology



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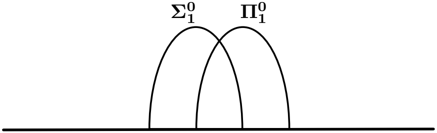

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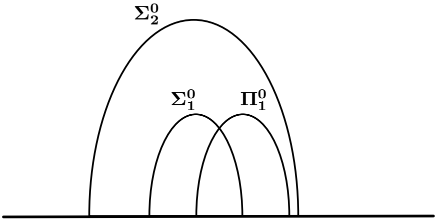

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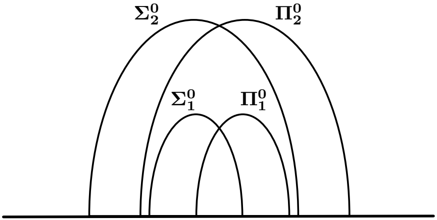

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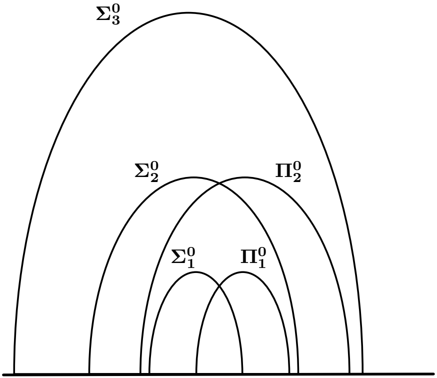
$\boldsymbol{\Sigma_3^0}$
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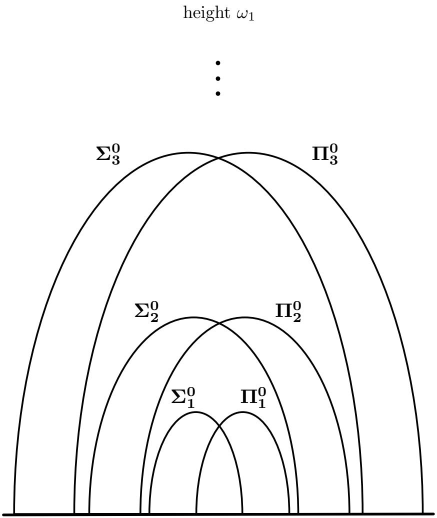
height ω_1
$\vdots$
$\mathbf{\Sigma^0_3}$
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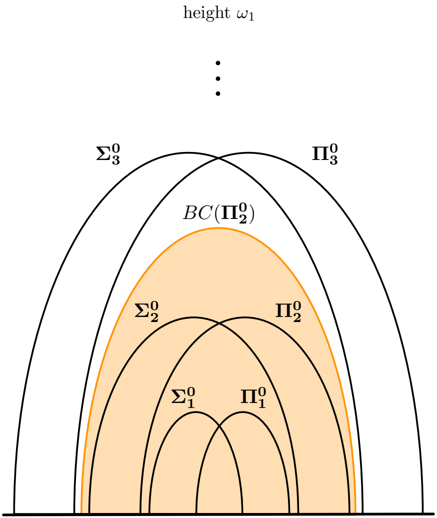
height ω_1
⋮
$\mathbf{\Sigma_3^0}$
$\mathbf{\Pi_3^0}$
$BC(\mathbf{\Pi_2^0})$
$\mathbf{\Sigma_2^0}$
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$\mathbf{\Sigma_1^0}$
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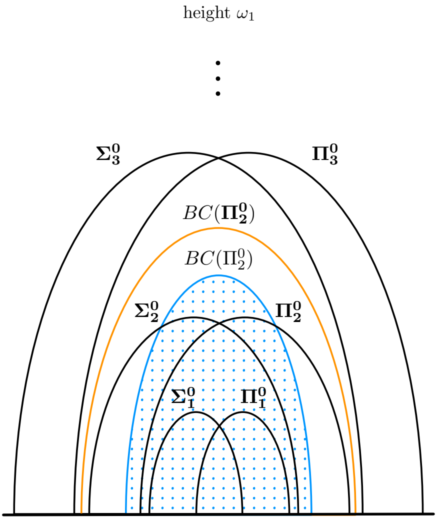
Topology



height ω_1
⋮
$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Pi}^0_3$
$BC(\boldsymbol{\Pi}^0_2)$
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$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Sigma}^0_1$
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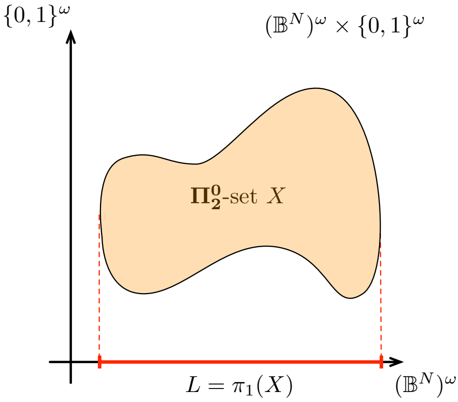
Topology

$$\begin{tikzcd}[column sep=small]
& \boldsymbol{\Delta}^1_1 & \\
& \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
& \boldsymbol{\Delta}^0_3 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma}^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_2 \arrow[ul, phantom, sloped, "\subsetneq"] \\
& \boldsymbol{\Delta}^0_2 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma}^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_1 \arrow[ul, phantom, sloped, "\subsetneq"] \\
& \boldsymbol{\Delta}^0_1 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}$$

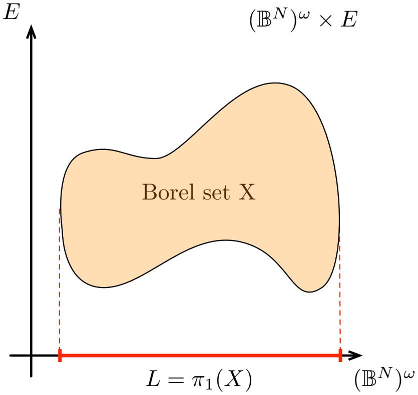
Topology

- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\boldsymbol{\Sigma}_1^1$) iff it is the first projection of some $\boldsymbol{\Pi}_2^0$-set $X \subseteq (\mathbb{B}^N)^\omega \times \{0,1\}^\omega$.



Topology

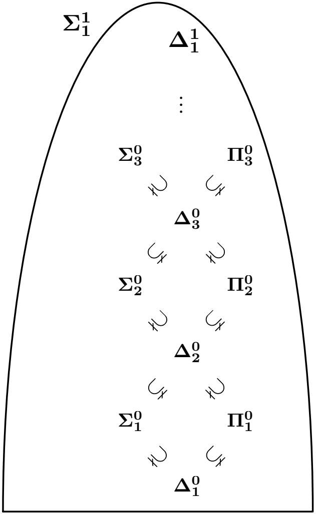
- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\mathbf{\Sigma}^1_1$) iff it is the first projection of some Borel set $X \subseteq (\mathbb{B}^N)^\omega \times E$, where E is a Polish space.



Topology

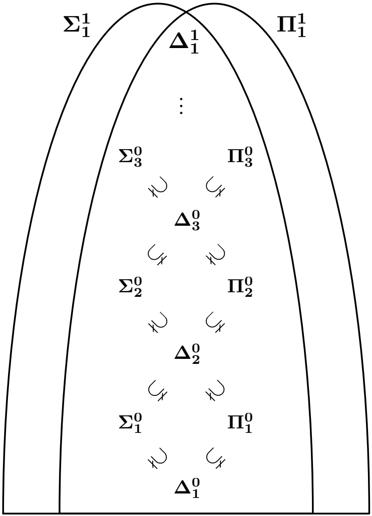
$$\begin{tikzcd}[column sep=small]
& \boldsymbol{\Delta}^1_1 & \\
& \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
& \boldsymbol{\Delta}^0_3 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma}^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_2 \arrow[ul, phantom, sloped, "\subsetneq"] \\
& \boldsymbol{\Delta}^0_2 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma}^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_1 \arrow[ul, phantom, sloped, "\subsetneq"] \\
& \boldsymbol{\Delta}^0_1 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}$$

Topology

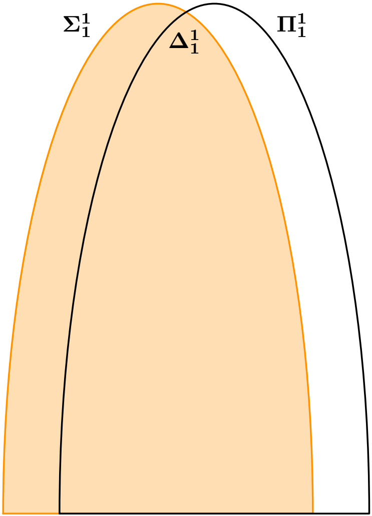


$\boldsymbol{\Sigma}^1_1$
$\boldsymbol{\Delta}^1_1$
$\vdots$
$\boldsymbol{\Sigma}^0_3$ $\boldsymbol{\Pi}^0_3$
$\boldsymbol{\Delta}^0_3$
$\boldsymbol{\Sigma}^0_2$ $\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Delta}^0_2$
$\boldsymbol{\Sigma}^0_1$ $\boldsymbol{\Pi}^0_1$
$\boldsymbol{\Delta}^0_1$

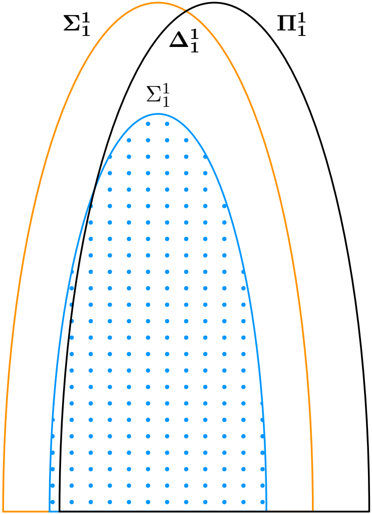
Topology


$\boldsymbol{\Sigma}^1_1$
$\boldsymbol{\Pi}^1_1$
$\boldsymbol{\Delta}^1_1$
$\vdots$
$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Pi}^0_3$
$\boldsymbol{\Delta}^0_3$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Delta}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$
$\boldsymbol{\Delta}^0_1$

Topology

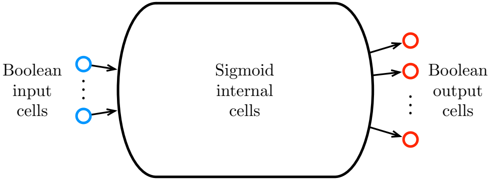


Topology



Deterministic ω-RNNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



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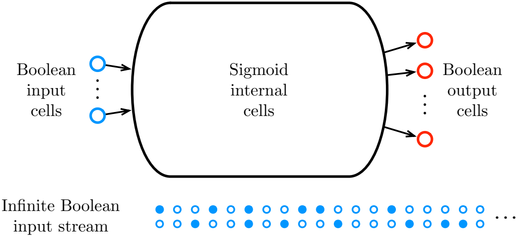

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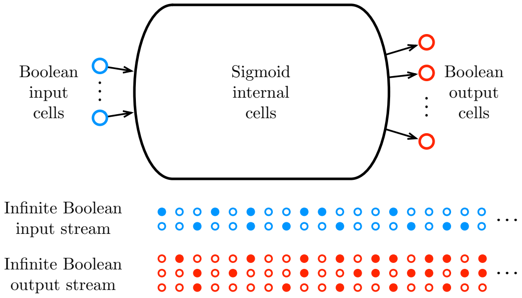

Deterministic ω-RNNs

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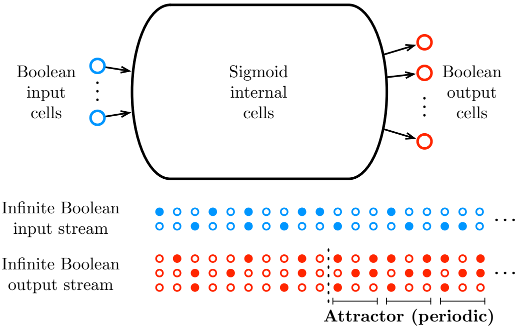
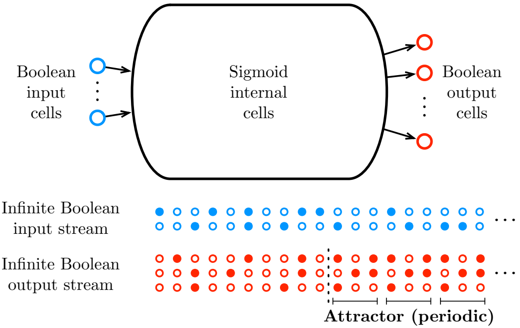


- The attractors are assumed to be classified into two possible kinds: *accepting* or *rejecting*.

Deterministic ω-RNNs

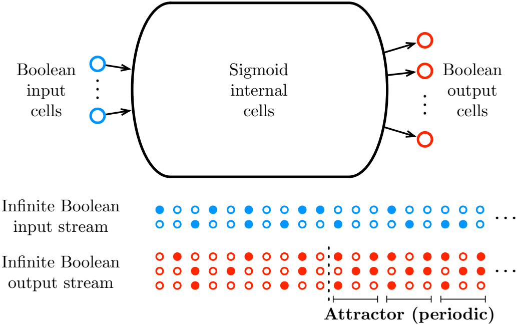
We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- An infinite Boolean input stream is *accepted* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *accepting* attractor.

Deterministic ω-RNNs

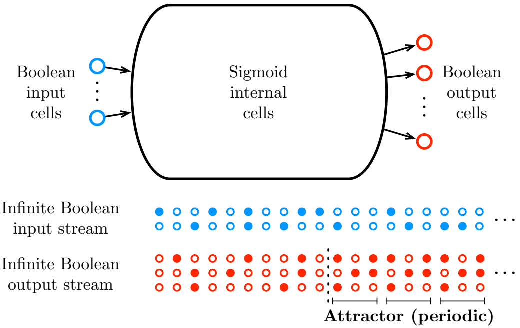
We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- An infinite Boolean input stream is *rejected* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *rejecting* attractor.

Deterministic ω-RNNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- The set of all input streams that are accepted by $\mathcal{N}$ is the *ω-language* recognized by $\mathcal{N}$.

Deterministic ω-RNNs

We consider six models of deterministic RNNs:

1. static rational RNNs: D-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s

Deterministic ω-RNNs

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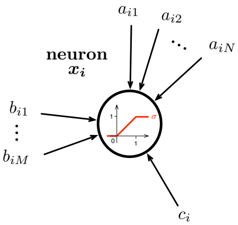
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



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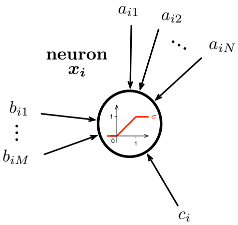
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



Deterministic ω-RNNs

We consider six models of deterministic RNNs:

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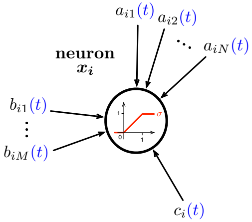
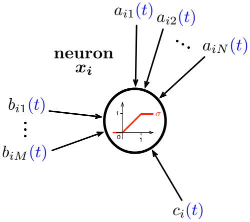
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



Deterministic ω-RNNs

We consider six models of deterministic RNNs:

1. static rational RNNs: D-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



Deterministic ω-RNNs

We consider six models of deterministic RNNs:

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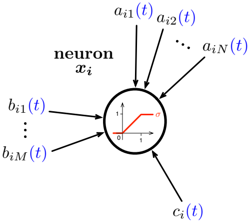
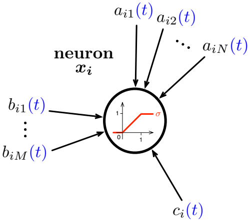
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



Deterministic ω-RNNs

We consider six models of deterministic RNNs:

1. static rational RNNs: D-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



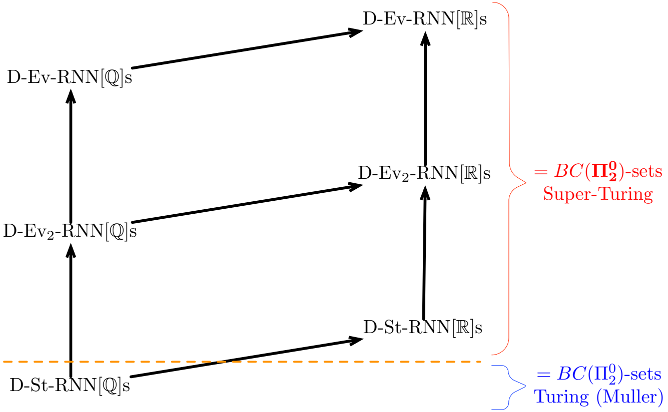
Results

Relationship between the models:

$$\begin{tikzcd}
 & \text{D-Ev-RNN}[\mathbb{R}]\text{s} \\
\text{D-Ev-RNN}[\mathbb{Q}]\text{s} \arrow[ur] & \\
 & \text{D-Ev}_2\text{-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{D-Ev}_2\text{-RNN}[\mathbb{Q}]\text{s} \arrow[ur] \arrow[uu] & \\
 & \text{D-St-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{D-St-RNN}[\mathbb{Q}]\text{s} \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Results

Relationship between the models:



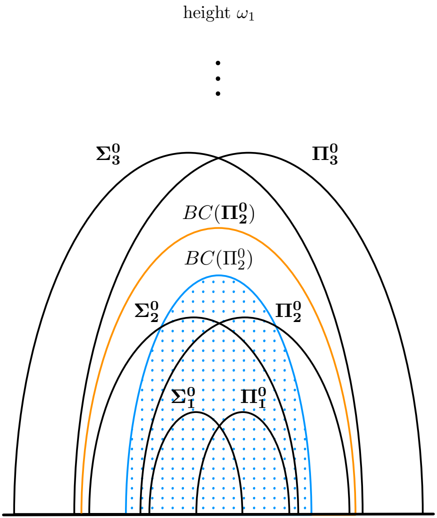
Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\Pi^0_2)$
- *L is recognizable by some deterministic Muller TM*
- *L is recognizable by some D-St-RNN[$\mathbb{Q}$]*

Results



height ω_1
$\vdots$
$\mathbf{\Sigma^0_3}$
$\mathbf{\Pi^0_3}$
$BC(\mathbf{\Pi^0_2})$
$BC(\Pi^0_2)$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

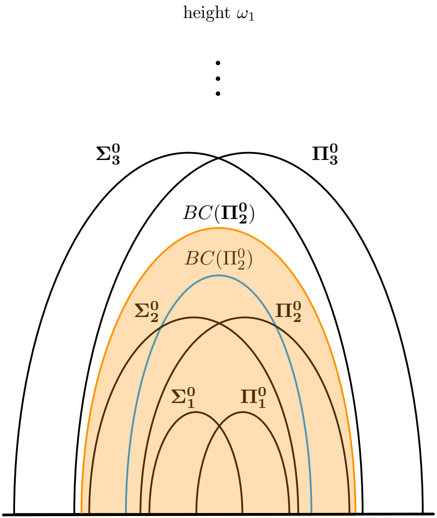
Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\mathbf{\Pi^0_2})$;
- L *is recognizable by some D-St-RNN[$\mathbb{R}$];*
- L *is recognizable by some D-Ev$_2$-RNN[$\mathbb{Q}$];*
- L *is recognizable by some D-Ev-RNN[$\mathbb{Q}$];*
- L *is recognizable by some D-Ev$_2$-RNN[$\mathbb{R}$];*
- L *is recognizable by some D-Ev-RNN[$\mathbb{R}$].*

Results


height ω_1
$\vdots$
$\mathbf{\Sigma^0_3}$
$\mathbf{\Pi^0_3}$
$BC(\mathbf{\Pi^0_2})$
$BC(\Pi^0_2)$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

Results – Summary

DET.	Static	Bi-valued Evolving	General Evolving
$\mathbb{Q}$	D-St-RNN[$\mathbb{Q}$]s $= BC(\Pi^0_2)$ Turing (Muller)	D-Ev$_2$-RNN[$\mathbb{Q}$]s $= BC(\mathbf{\Pi^0_2})$ super-Turing	D-Ev-RNN[$\mathbb{Q}$]s $= BC(\mathbf{\Pi^0_2})$ super-Turing
$\mathbb{R}$	D-St-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi^0_2})$ super-Turing	D-Ev$_2$-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi^0_2})$ super-Turing	D-Ev-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi^0_2})$ super-Turing

Deterministic ω-RNNs

We consider two other models of deterministic RNNs:

1. static real RNNs with only one real weight:

   $$\text{D-St-RNN}[\mathbb{Q}, \ulcorner \alpha \urcorner]\text{s, where } \ulcorner \alpha \urcorner \in \mathbb{R} \text{ encoding of } \alpha \in \{0,1\}^\omega$$

2. bi-valued evolving rational RNNs with only one evolving weight:

   $$\text{D-Ev}_2\text{-RNN}[\mathbb{Q}, \alpha]\text{s, where } \alpha \in \{0,1\}^\omega$$

Deterministic ω-RNNs

We consider two other models of deterministic RNNs:

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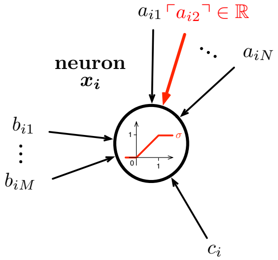
   D-St-RNN[$\mathbb{Q}, \ulcorner\alpha\urcorner$]s, where $\ulcorner\alpha\urcorner \in \mathbb{R}$ encoding of $\alpha \in \{0,1\}^\omega$

2. bi-valued evolving rational RNNs with only one evolving weight:

   D-Ev$_2$-RNN[$\mathbb{Q}, \alpha$]s, where $\alpha \in \{0,1\}^\omega$



Deterministic ω-RNNs

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1. static real RNNs with only one real weight:

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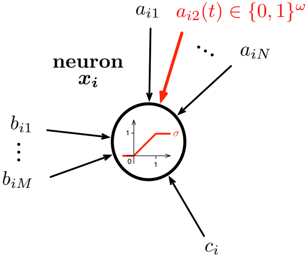
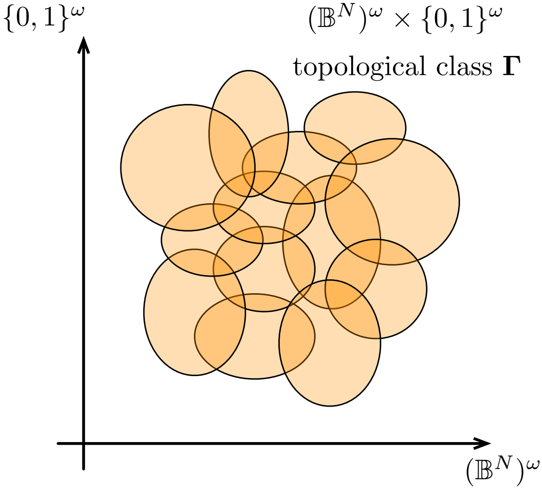
   D-Ev$_2$-RNN$[\mathbb{Q}, \alpha]$s, where $\alpha \in \{0,1\}^\omega$

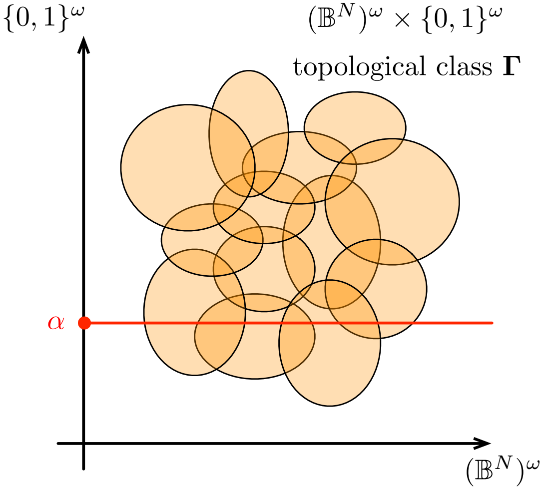


Topology

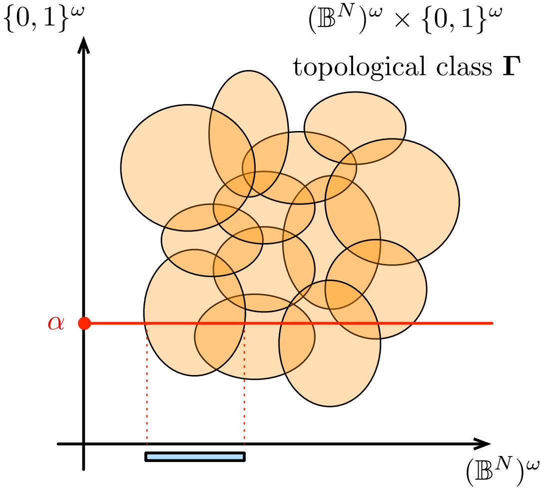


$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
$(\mathbb{B}^N)^\omega$

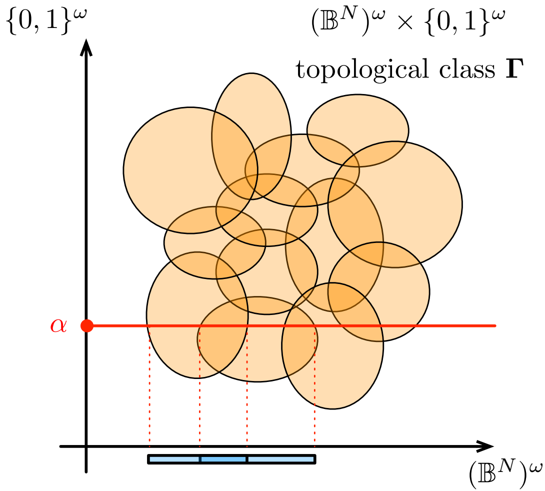
Topology


$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
α
$(\mathbb{B}^N)^\omega$

Topology

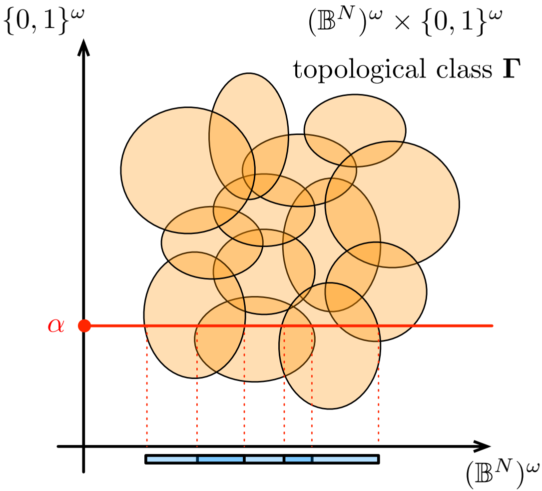

$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
α
$(\mathbb{B}^N)^\omega$

Topology

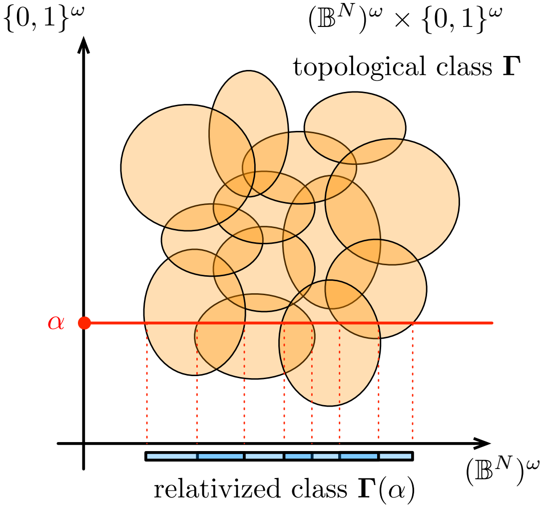


$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
α
$(\mathbb{B}^N)^\omega$

Topology


$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
α
$(\mathbb{B}^N)^\omega$

Topology



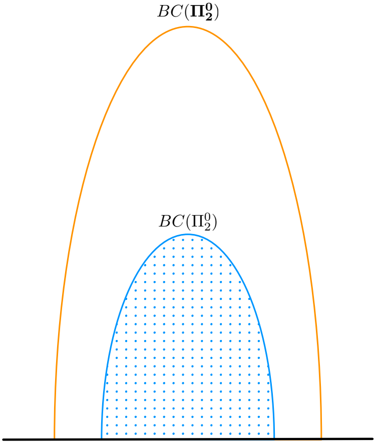
$\{0,1\}^\omega$
$(\mathbb{B}^N)^\omega \times \{0,1\}^\omega$
topological class $\mathbf{\Gamma}$
α
$(\mathbb{B}^N)^\omega$
relativized class $\mathbf{\Gamma}(\alpha)$

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\Pi^0_2)(\alpha)$, *for some* $\alpha \in \{0,1\}^\omega$
- L *is recognizable by some D-St-RNN*[$\mathbb{Q}, \ulcorner \alpha \urcorner$]
- L *is recognizable by some D-Ev$_2$-RNN*[$\mathbb{Q}, \alpha$]



Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

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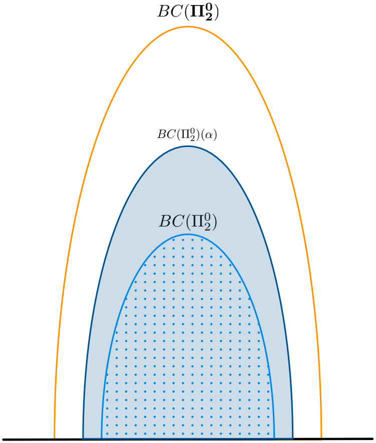
- L *is recognizable by some D-St-RNN*[$\mathbb{Q}, \ulcorner\alpha\urcorner$]
- L *is recognizable by some D-Ev*$_2$*-RNN*[$\mathbb{Q}, \alpha$]



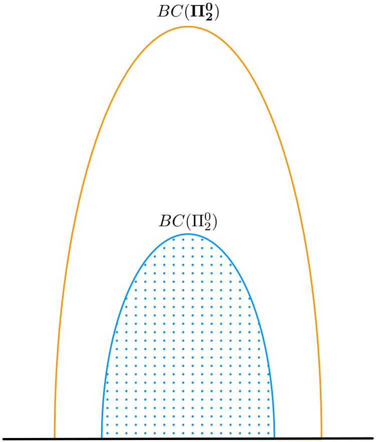
Results

Proposition

There exist some infinite sequence $(\alpha_k)_{k<\omega_1}$, where each $\alpha_i \in \{0,1\}^\omega$, such that

$$BC(\Pi^0_2)(\alpha_i) \subsetneq BC(\Pi^0_2)(\alpha_j)$$

for all $i < j < \omega_1$.



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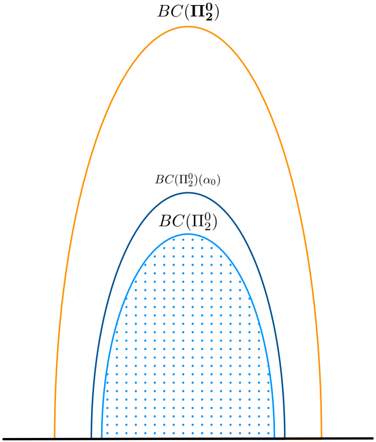

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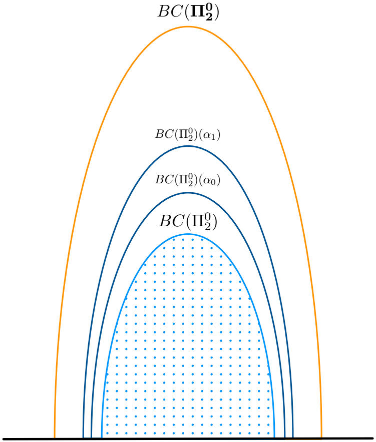
for all $i < j < \omega_1$.



Results

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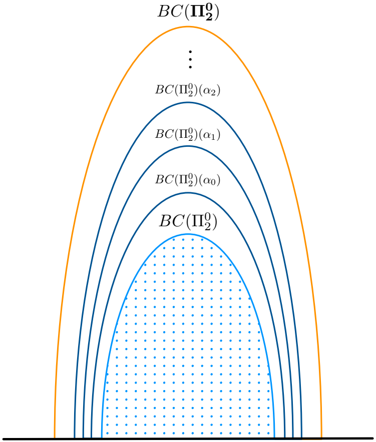
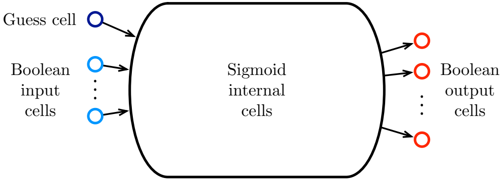


Nondeterministic ω-RNNs (Type I)

The RNNs are provided with an additional Boolean guess cell.



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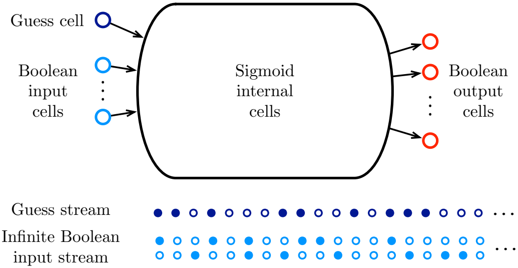

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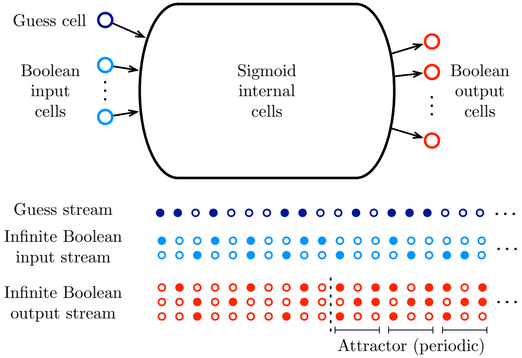

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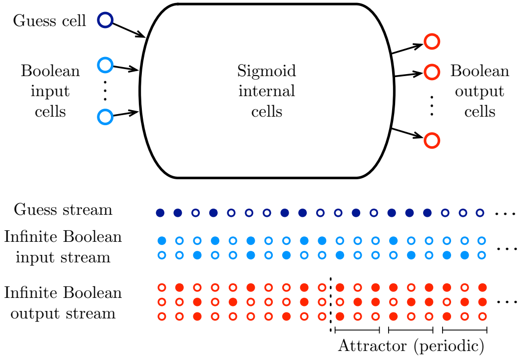

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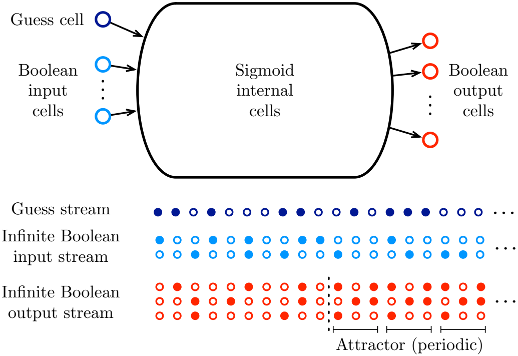

- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ enters an accepting attractor.

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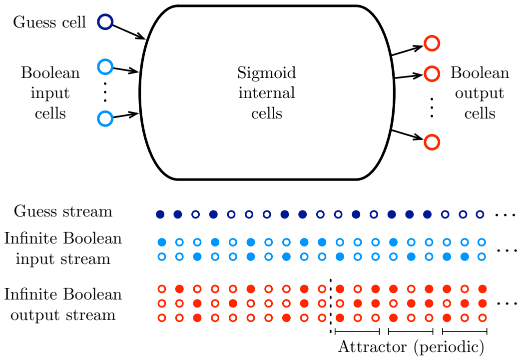

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Nondeterministic ω-RNNs (Type I)

We consider six models of nondeterministic RNNs (of type I):

1. static rational RNNs: N-St-RNN[$\mathbb{Q}$]s
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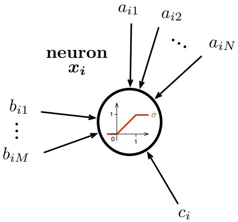

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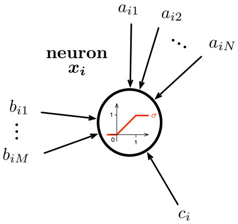

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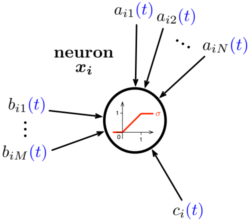

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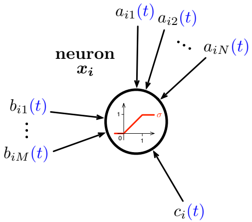

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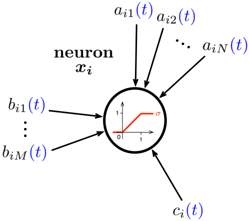

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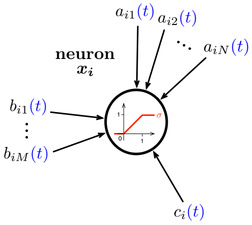
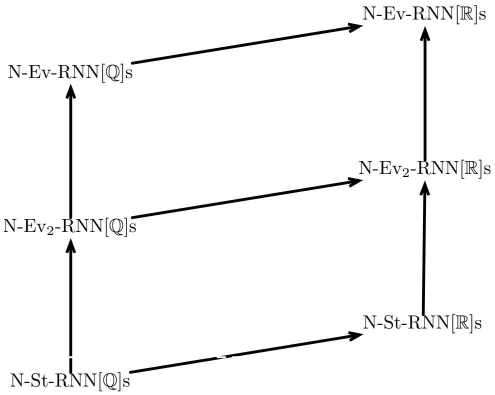


Results

Relationship between the models:



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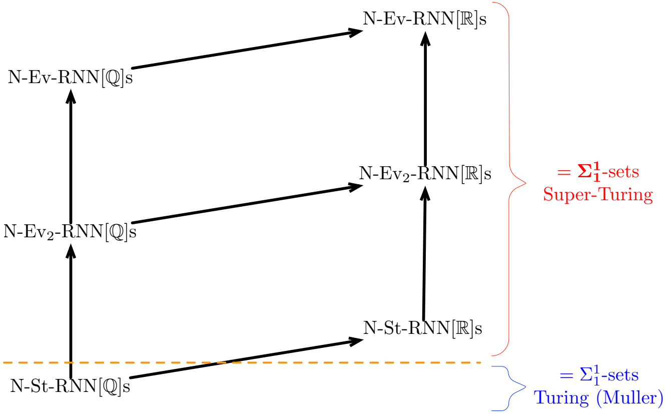


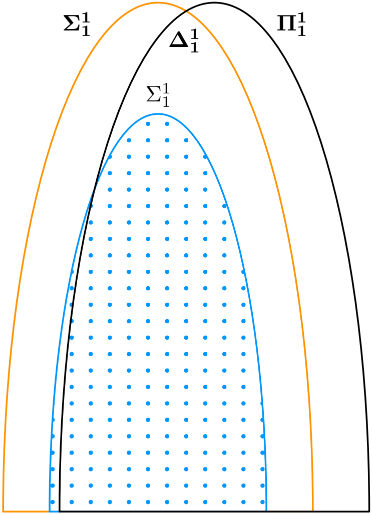
Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \Sigma^1_1$
- L *is recognizable by some nondeterministic Muller TM*
- L *is recognizable by some N-St-RNN[$\mathbb{Q}$]*

Results



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- $L \in \Sigma^1_1$;
- L *is recognizable by some N-St-RNN[$\mathbb{R}$];*
- L *is recognizable by some N-Ev$_2$-RNN[$\mathbb{Q}$];*
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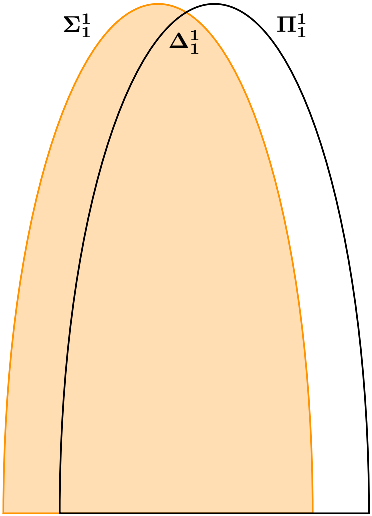

Results – Summary

NONDET.	Static	Bi-valued Evolving	General Evolving
$\mathbb{Q}$	N-St-RNN[$\mathbb{Q}$]s $= \Sigma^1_1$ Turing (Muller)	N-Ev$_2$-RNN[$\mathbb{Q}$]s $= \boldsymbol{\Sigma}^1_1$ super-Turing	N-Ev-RNN[$\mathbb{Q}$]s $= \boldsymbol{\Sigma}^1_1$ super-Turing
$\mathbb{R}$	N-St-RNN[$\mathbb{R}$]s $= \boldsymbol{\Sigma}^1_1$ super-Turing	N-Ev$_2$-RNN[$\mathbb{R}$]s $= \boldsymbol{\Sigma}^1_1$ super-Turing	N-Ev-RNN[$\mathbb{R}$]s $= \boldsymbol{\Sigma}^1_1$ super-Turing

Nondeterministic ω-RNNs (Type I)

We consider two other models of nondeterministic RNNs (type I):

1. static real RNNs with only one real weight:

   N-St-RNN[$\mathbb{Q}, \ulcorner\alpha\urcorner$]s, where $\ulcorner\alpha\urcorner \in \mathbb{R}$ encoding of $\alpha \in \{0,1\}^\omega$

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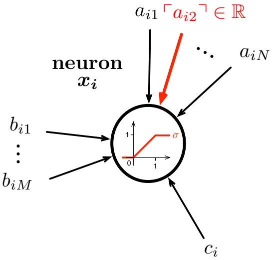

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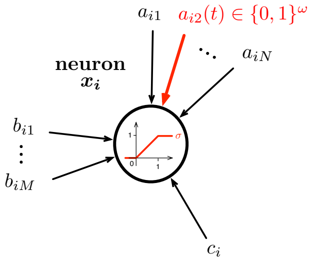

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Theorem

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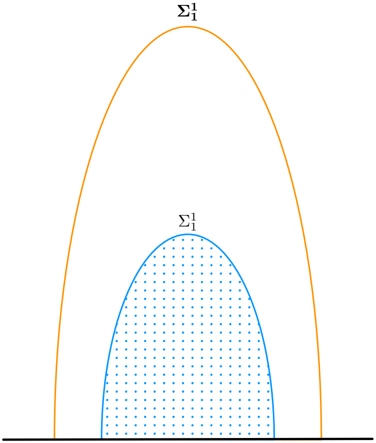

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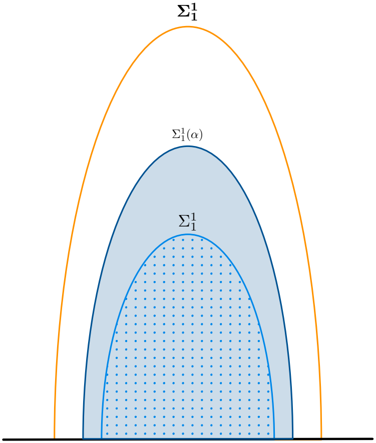

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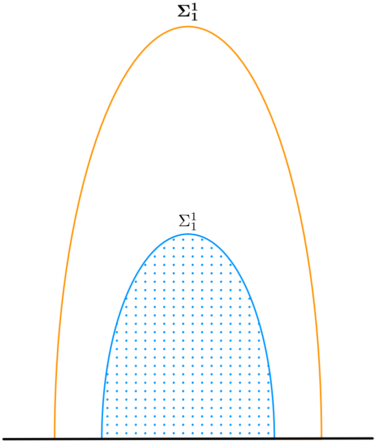
Proposition

There exist some infinite sequence $(\alpha_k)_{k<\omega_1}$, where each $\alpha_i \in \{0,1\}^\omega$, such that

$$\Sigma^1_1(\alpha_i) \subsetneq \Sigma^1_1(\alpha_j)$$

for all $i < j < \omega_1$.



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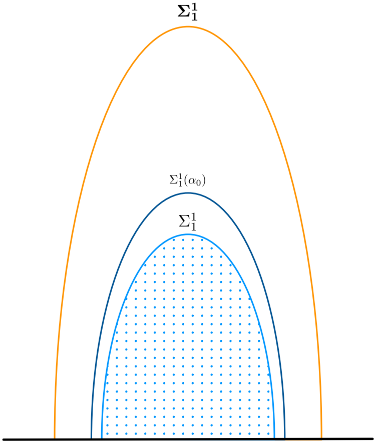

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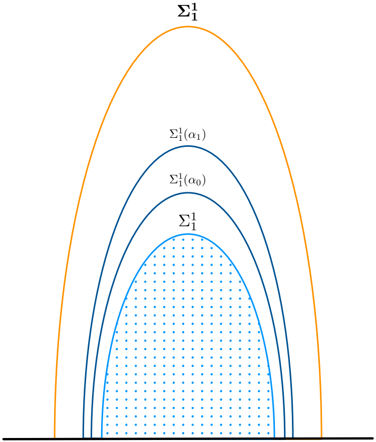

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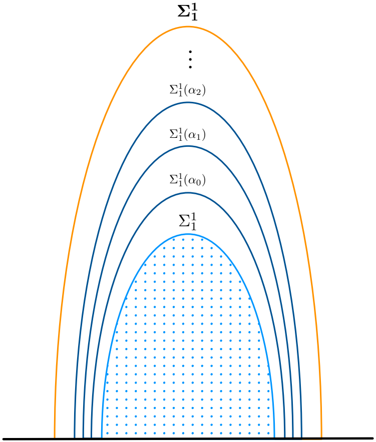

Nondeterministic ω-RNNs (Type II)

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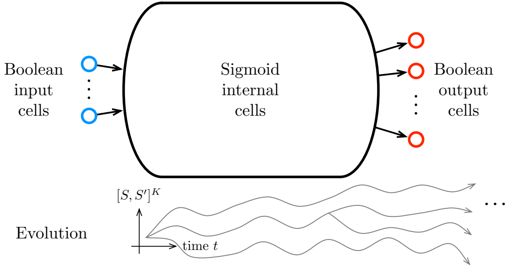

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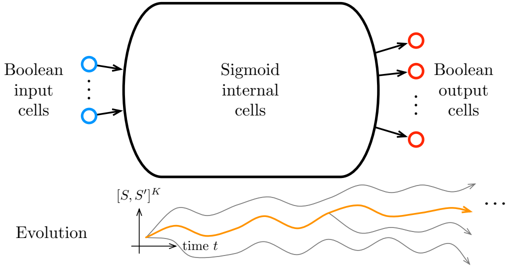

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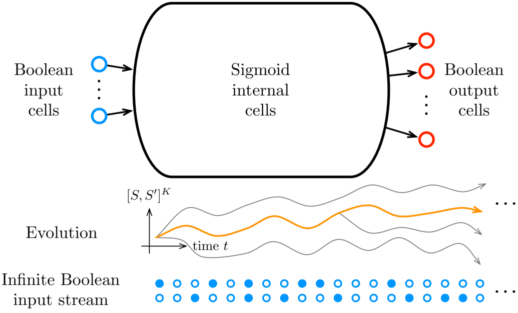

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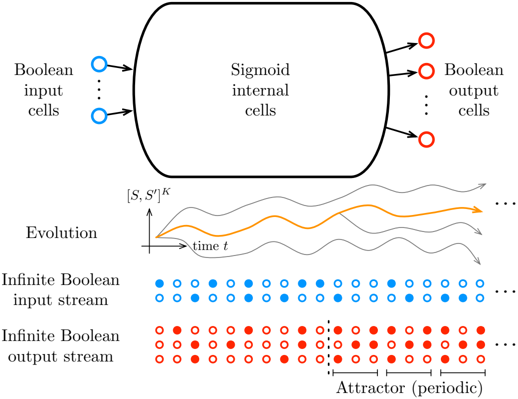

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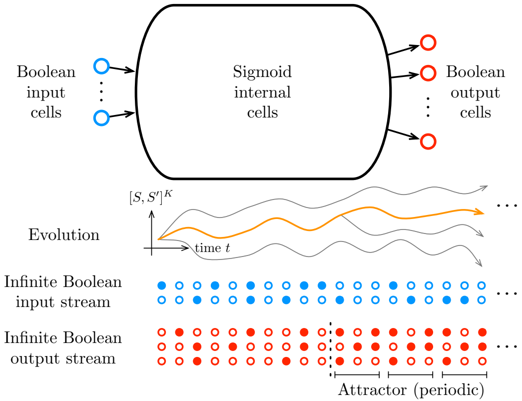

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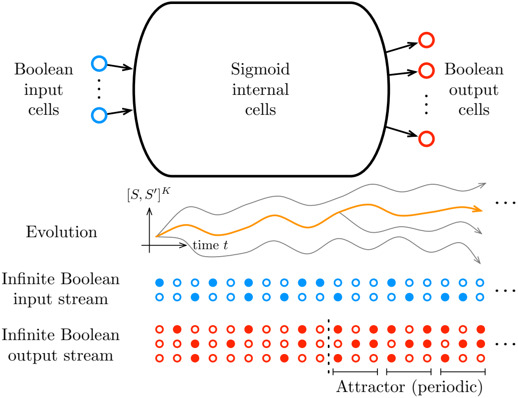

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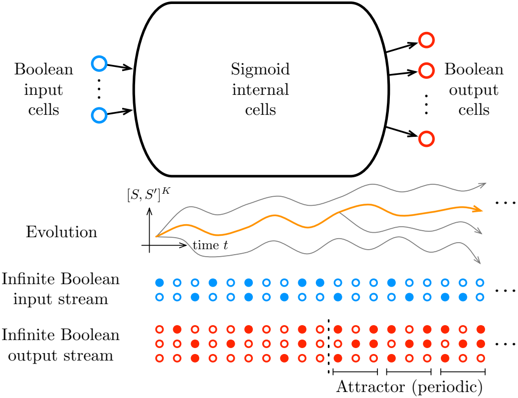

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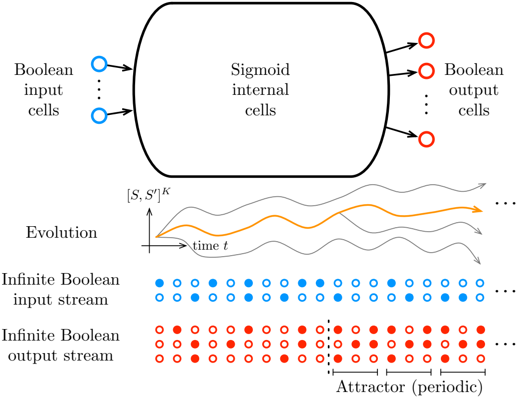

- We have more results in this context also...

Conclusion

- We provided a characterization of the expressive power of recurrent neural networks in terms of their attractor dynamics.
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