Constructing the Algebraic Counterpart of the Wagner Hierarchy by Way of Games

joint work with Jacques Duparc

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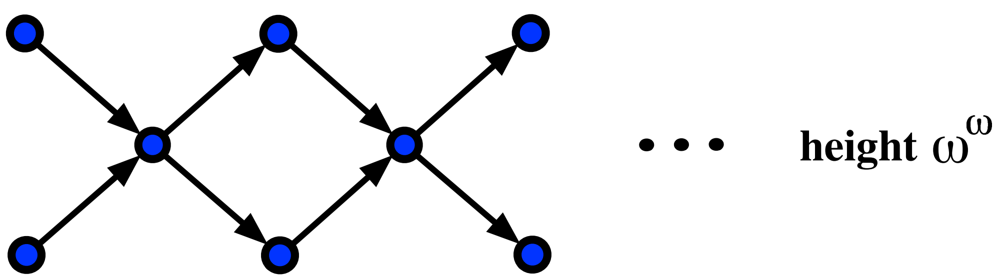
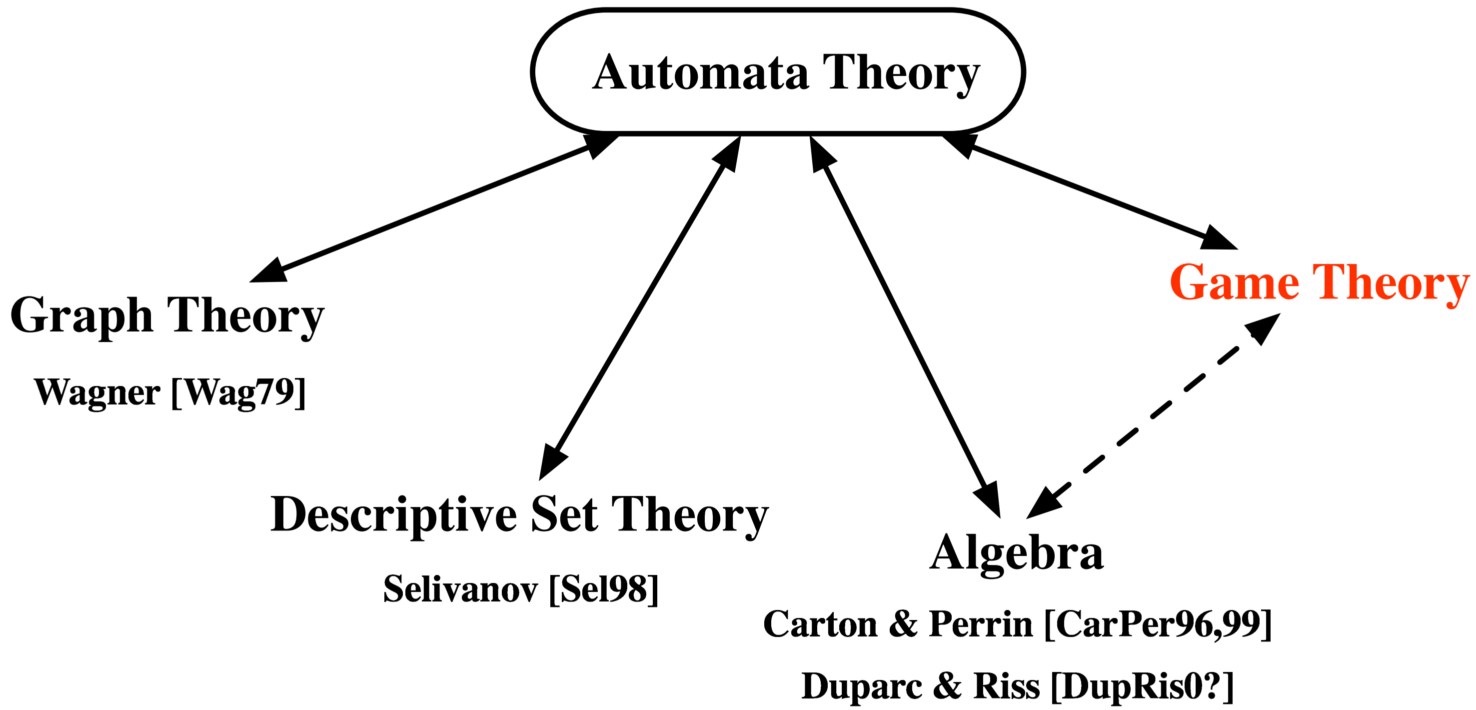
The Wagner Hierarchy
(hierarchy of ω-rational sets)

Let $\mathcal{A}$ and $\mathcal{B}$ be two Muller automata,

$$\boldsymbol{L(\mathcal{A}) \leq_W L(\mathcal{B})} \Leftrightarrow_{def} \exists\ f \text{ continous s.t.}$$
$$L(\mathcal{A}) = f^{-1}\big(L(\mathcal{B})\big)$$



Studies of the Wagner hierarchy



We give a natural game theoretical description...

AUTOMATA THEORY		ALGEBRA
automaton (rational language)	$\longleftrightarrow$	finite semigroup
Büchi automaton (ω-rational language)	$\longleftrightarrow$	Wilke algebra finite ω-semigroup

ω-*semigroup* $S = (S_+, S_\omega)$ (J.-É. Pin)

- $(S_+, \cdot)$ is a semigroup, S_ω is a set
- $\pi : {S_+}^\omega \longrightarrow S_\omega$ an infinite product

We consider finite ω-semigroups $S = (S_+, S_\omega)$ of the form:

- S_+ **is a finite semigroup**
- $S_\omega = \left\{ \overline{(s,e)} : (s,e) \text{ **is a linked pair of** } S_+ \right\}$

A reduction relation $\leq_{SG}$ **on** ω**-semigroups**

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$

$\boldsymbol{X \leq_{SG} Y}$ $\Leftrightarrow_{def}$ X **is "less complicated" than** Y

i.e. $\exists$ **"simple"** f **s.t.** $(u \in X \Leftrightarrow f(u) \in Y)$

$\Leftrightarrow_{def}$ **II has a w.s. in the game** $\mathbb{SG}(X, Y)$

An infinite two-player game $\mathbb{SG}(\boldsymbol{X}, \boldsymbol{Y})$ *on* ω*-semigroups*

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$**.**

(X)	**I**	s_0	s_1	$\ldots$	$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle s_0, s_1, s_2, \ldots \rangle$
(Y)	**II**	t_0	t_1	$\ldots$	$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle t_0, t_1, t_2, \ldots \rangle$

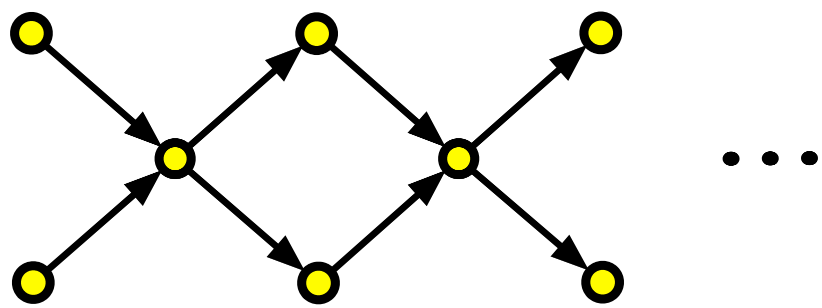
II wins

$$\Leftrightarrow_{def}$$

$$\boldsymbol{\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y}$$

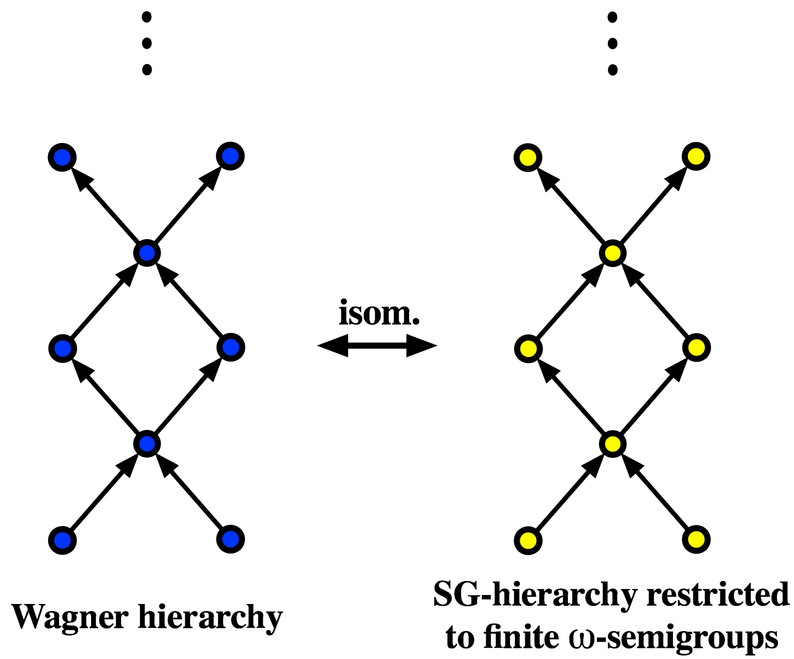
The $\mathbb{SG}$-hierarchy

The $\leq_{SG}$ induces a hierarchy on ω-subsets



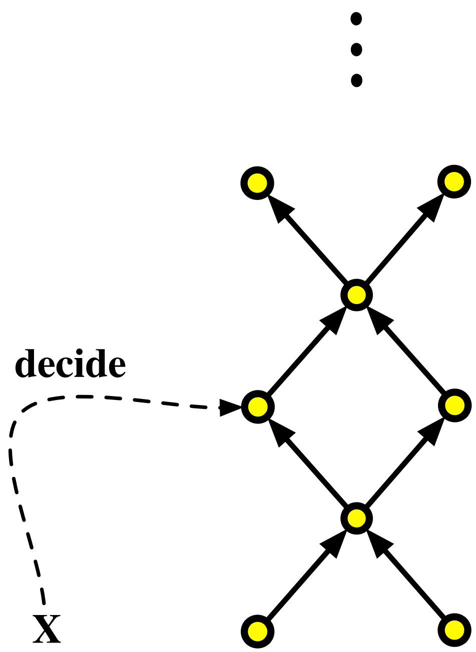
Proposition

The finite $\mathbb{SG}$-hierarchy is classwise isomorphic to the Wagner hierarchy.



Proposition

The finite $\mathbb{SG}$-hierarchy is decidable: given $X \subseteq S_\omega$, we can compute its degree ξ_X in the $\mathbb{SG}$-hierarchy



Linked pairs

Let S_+ be a semigroup,

$(s, e) \in S_+ \times S_+$ **is a *linked pair* if**

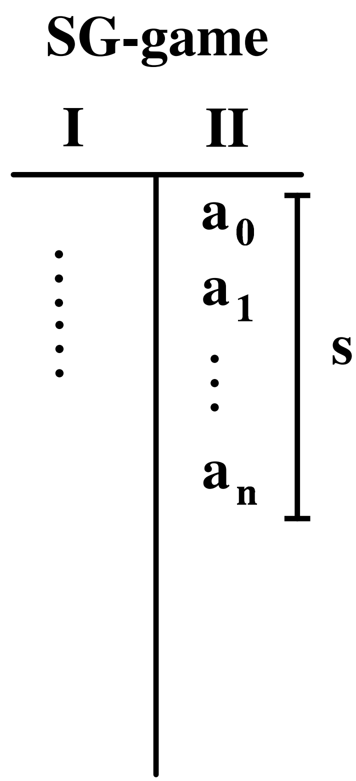
1. $se = s$
2. e **is idempotent (i.e.** $e^2 = e$**)**

s **is called the *prefix***

e **is called the *idempotent***

Linked pair $\equiv$ stable postion in $\mathbb{SG}$-game

Let (s, e) be a linked pair (so $se = s$ and $e^2 = e$),



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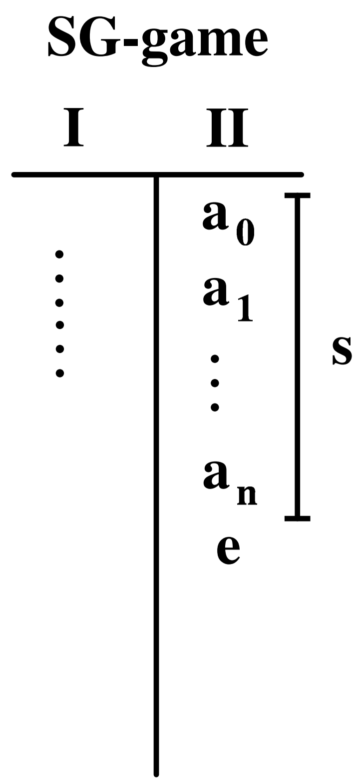

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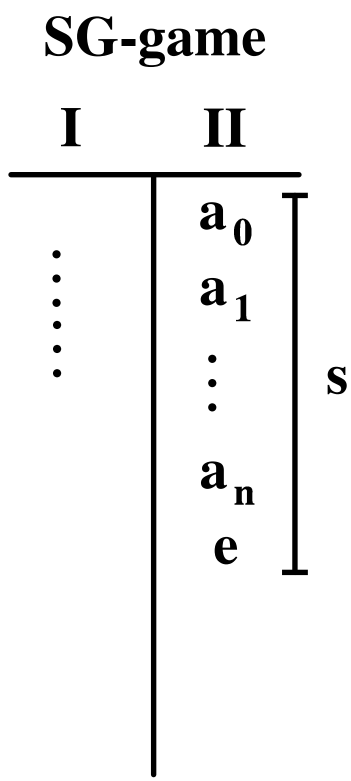

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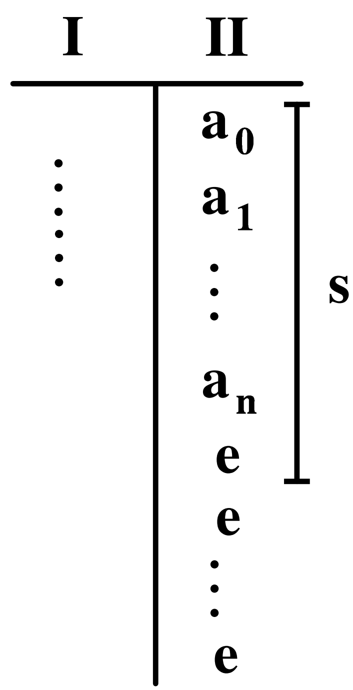
SG-game



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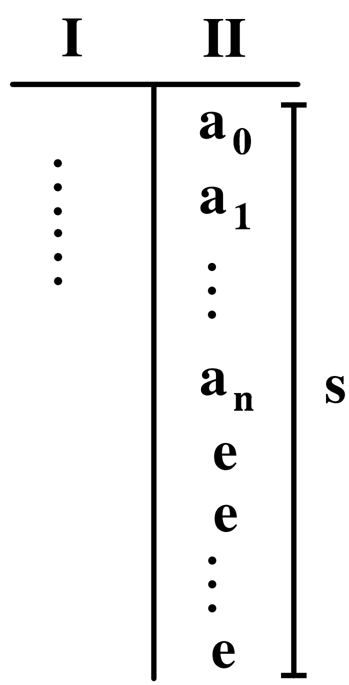

Accessibility relation $\leq_{pr}$ between prefixes of linked pairs

Let (s, e) **and** (s', e') **be two linked pairs,**

$$\boldsymbol{s} \leq_{\boldsymbol{pr}} \boldsymbol{s'} \quad \Leftrightarrow_{def} \quad \exists\ t \in S_+ \ \textbf{s.t.}\ st = s'$$

$$\Leftrightarrow_{def} \quad \textbf{a player can go from position } s \textbf{ to position } s' \textbf{ in a } \mathbb{SG}\textbf{-game}$$

and also...

$$\boldsymbol{s} \equiv_{\boldsymbol{pr}} \boldsymbol{s'} \quad \Leftrightarrow_{def} \quad s \leq_{pr} s' \leq_{pr} s$$

$$\bar{\boldsymbol{s}} =_{def} \{s' : s' \equiv_{pr} s\}$$

$$\bar{\boldsymbol{s}} \ \textcolor{blue}{\leq_{\boldsymbol{pr}}} \ \bar{\boldsymbol{s}}' \Leftrightarrow_{def} \exists\ s_i \in \bar{s}\ \exists\ s'_j \in \bar{s}' \ \textbf{s.t.}\ \ s_i \leq_{pr} s'_j$$

Lemma

Let S be a finite semigroup, then $(\overline{Pr}(S), \leq_{pr})$ is a partial ordering.

Absorption ordering $\leq_{id}$ **between idempotents of a same prefix**

Let e, e' **be any two idempotents,**

$$\boldsymbol{e} \leq_{\boldsymbol{id}} \boldsymbol{e'} \;\Leftrightarrow_{def}\; ee' = e'e = e'$$

Lemma

Let S be a finite semigroup, then $(E(S), \leq_{id})$ is a partial ordering.

Tree-representation of a finite semigroup

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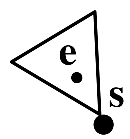
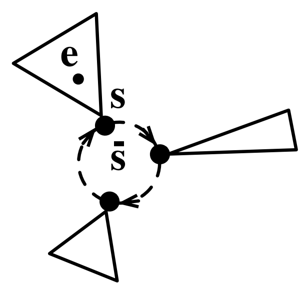


Tree-representation **of a finite semigroup**



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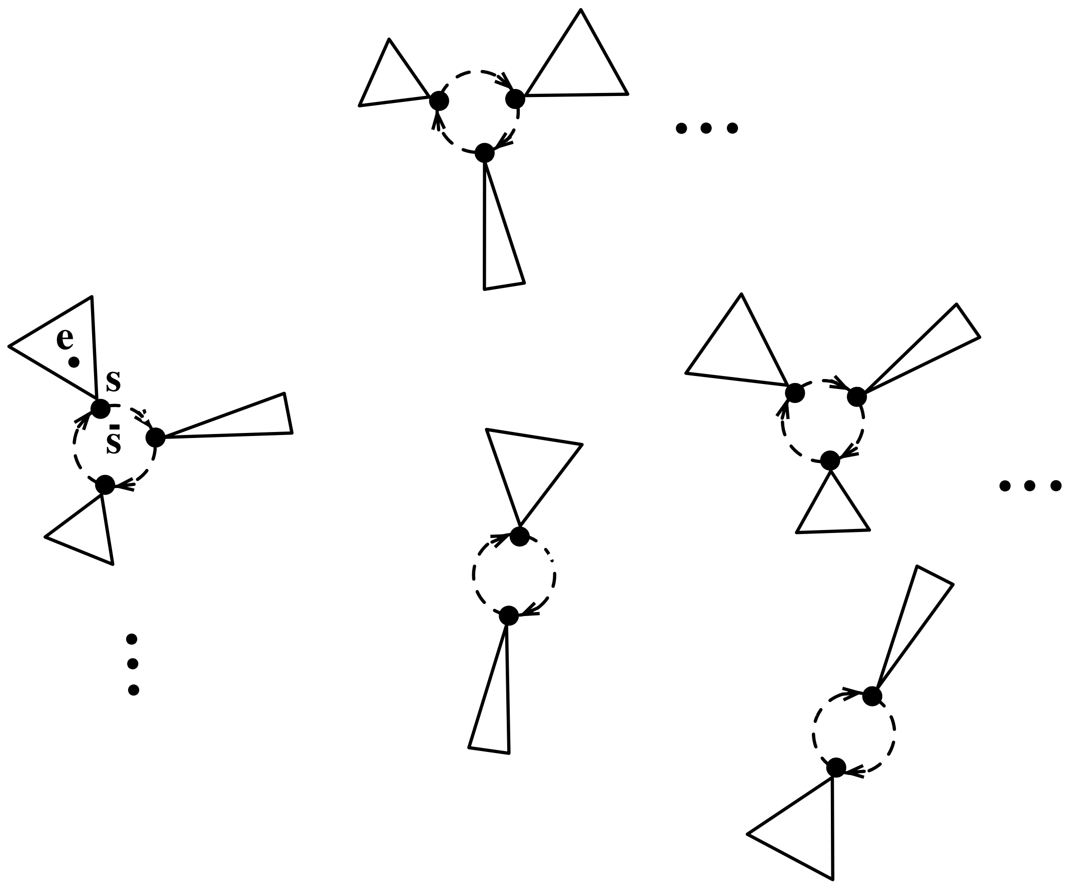

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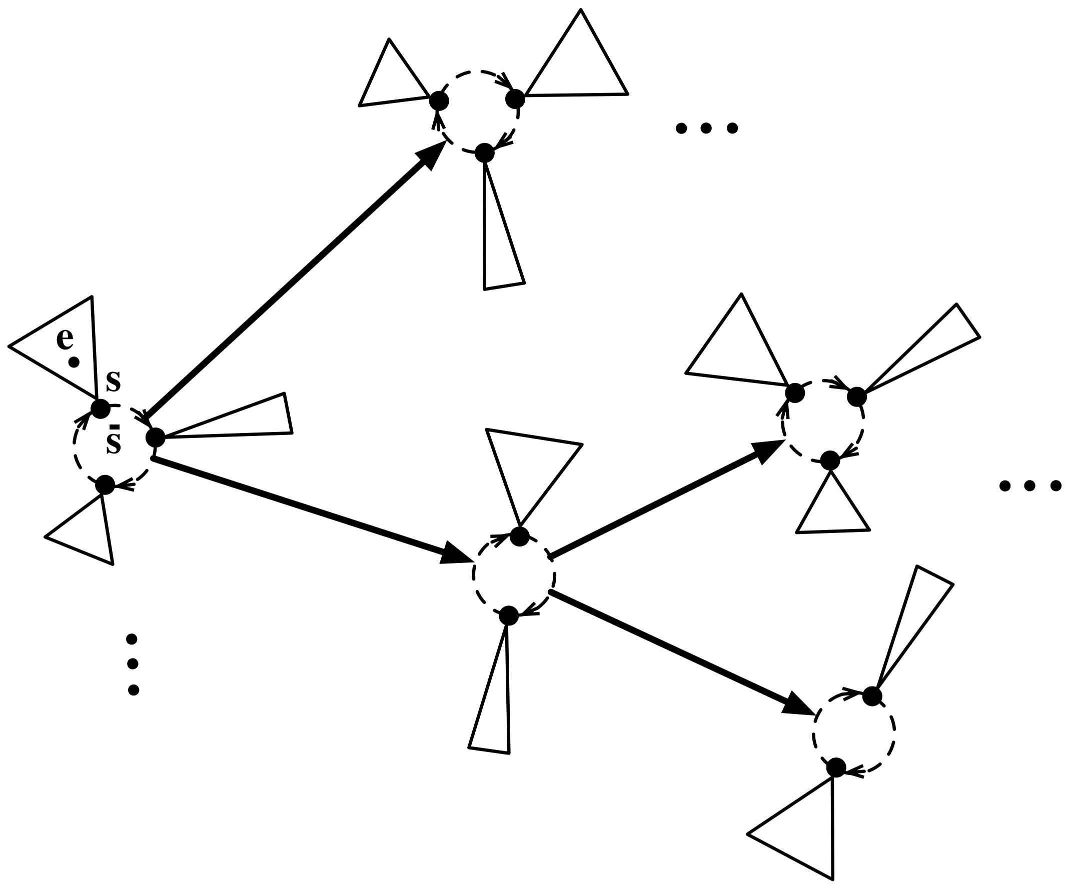
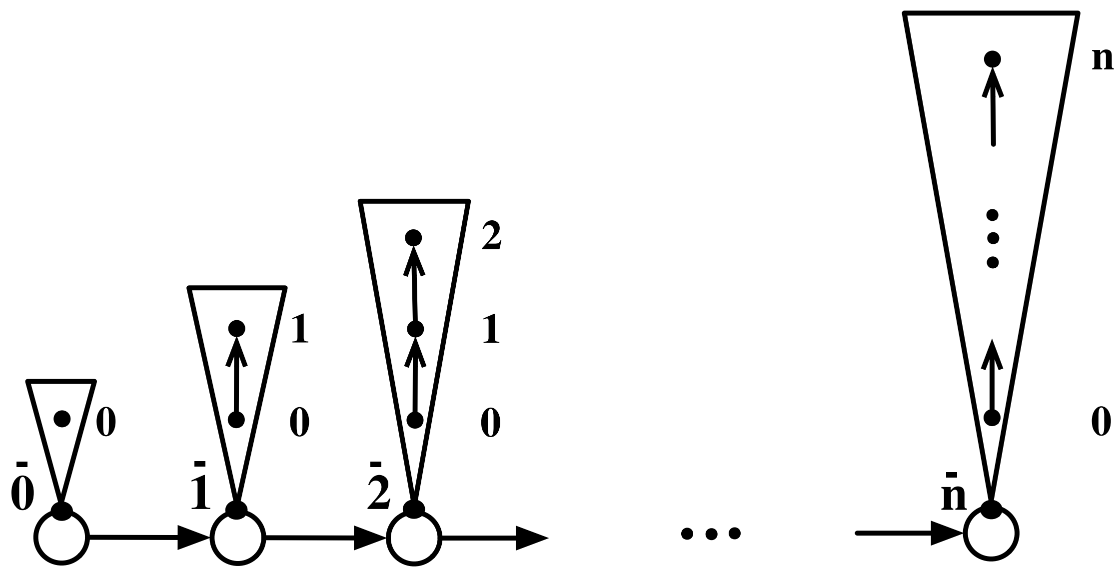


Example

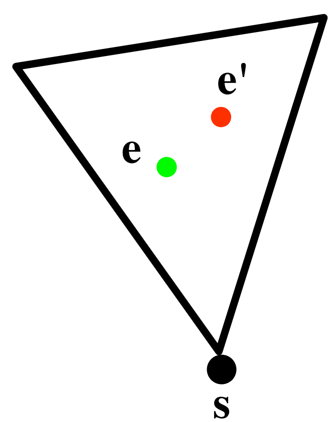
Let $S = (\{1, 2, \ldots, n\}, \max)$**, one has**



Coloring of linked pairs with respect to an ω-subset X

Let $S = (S_+, S_\omega)$ **be an** ω**-semigroup,** (s, e) **be a linked pair, and** $X \subseteq S_\omega$**,**

$$e \textbf{ in the petal } E_s(S_+) \textbf{ will be } \begin{cases} \textbf{green if } \overline{(s,e)} \in X \\ \textbf{red if } \overline{(s,e)} \notin X \end{cases}$$



Colored tree-representation of an ω-subset X

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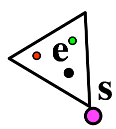

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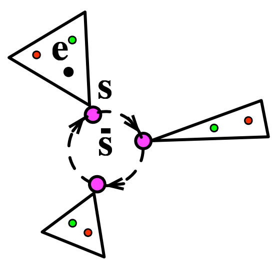

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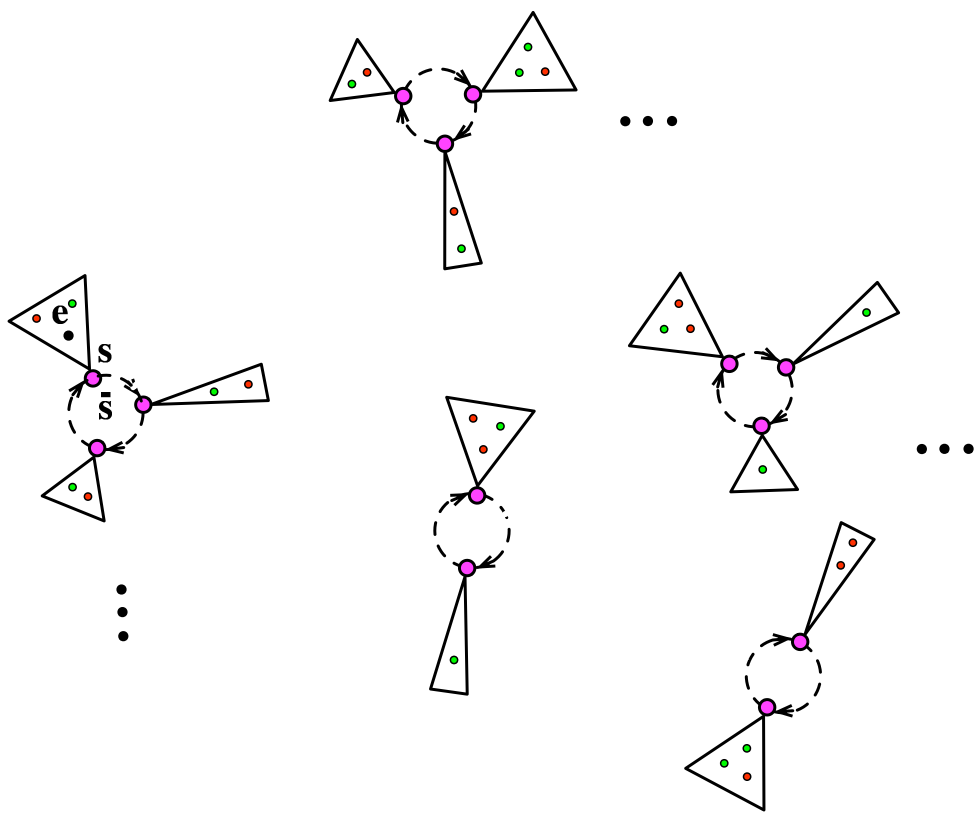

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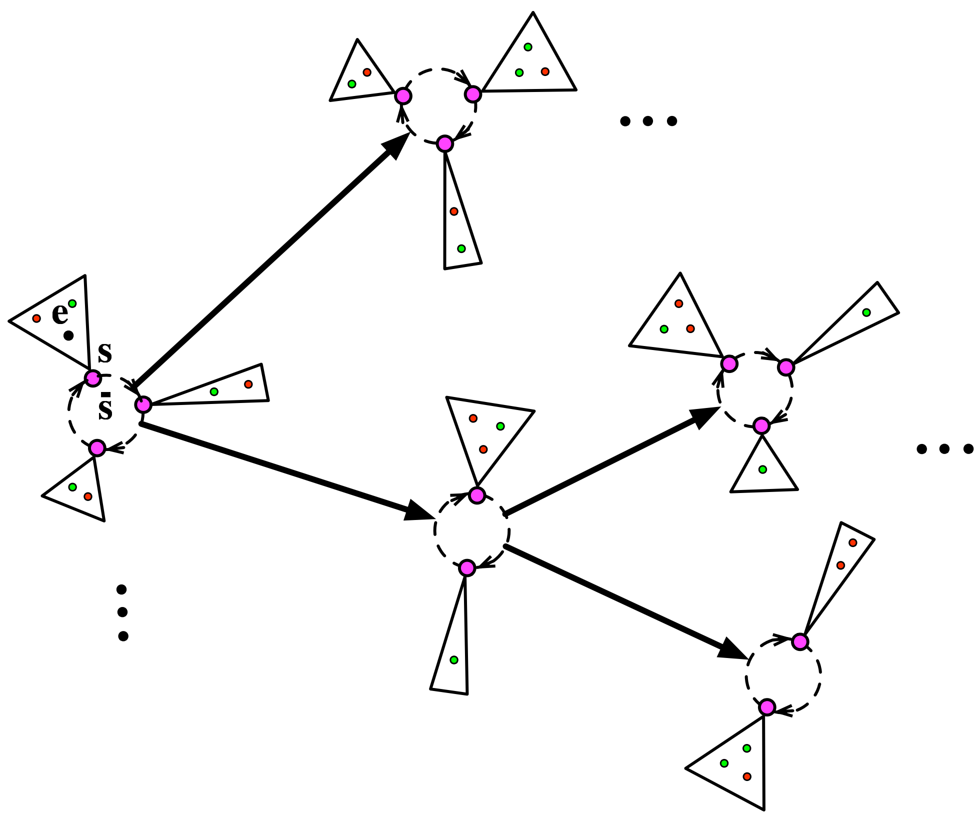
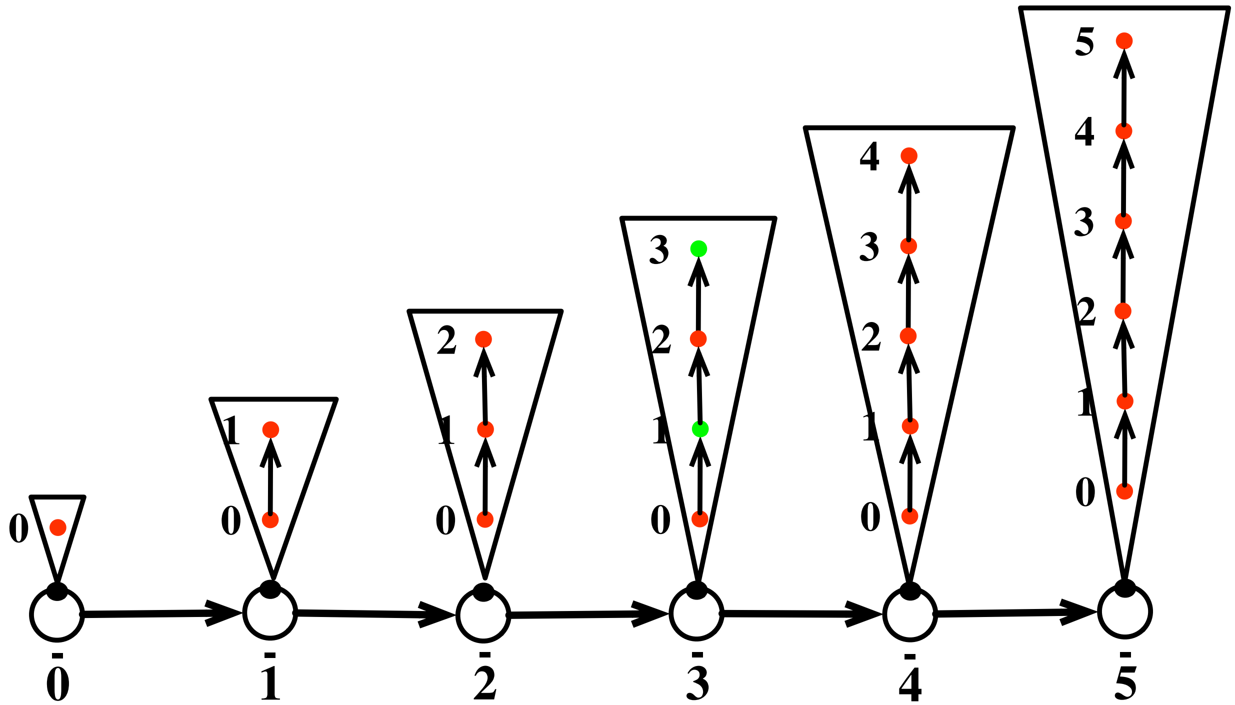


Algorithm

The algorithm deciding the $\mathbb{SG}$-degree of X follows from this colored tree-representation

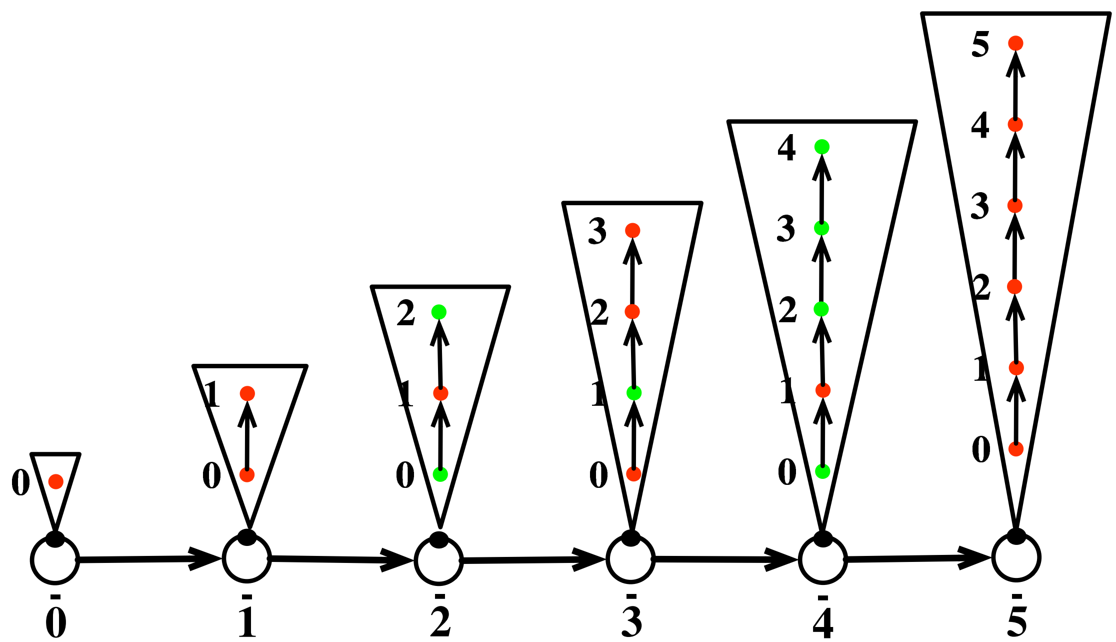
Example

Let $S = (S_+, S_\omega)$ **with** $S_+ = (\{0, 1, 2, 3, 4, 5\}, \max)$**, and** $X = \left\{\overline{(3,1)}, \overline{(3,3)}\right\}$**, then** $d^o_{sg}(X) = [-]\omega^3$**.**



Example

Let $S = (S_+, S_\omega)$ **with** $S_+ = (\{0,1,2,3,4,5\}, \max)$**, and** $X = \left\{\overline{(2,0)}, \overline{(2,2)}, \overline{(3,1)}, \overline{(4,0)}, \overline{(4,2)}, \overline{(4,3)}, \overline{(4,4)}\right\}$**, then** $d^o_{sg}(X) = [+]\omega^2 \cdot 3$**.**

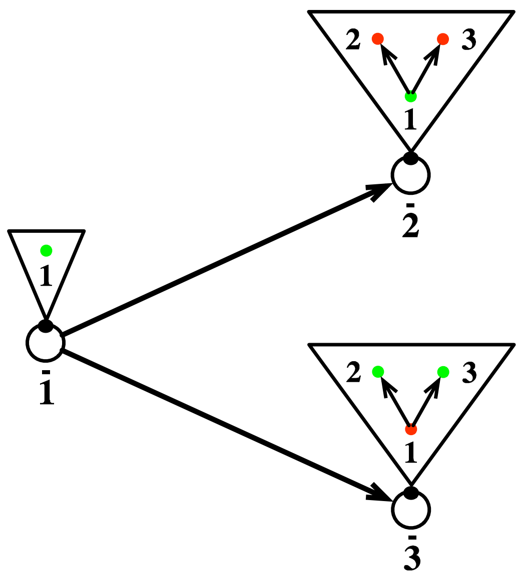


Example

Let $S = (S_+, S_\omega)$ **with** $S_+ = (\{1, 2, 3\}, \cdot)$ **where**

$\cdot$	1	2	3
1	1	2	3
2	2	2	2
3	3	3	3

and $X = \left\{\overline{(1,1)}, \overline{(2,1)}, \overline{(3,2)}, \overline{(3,3)}\right\}$**, then**
$d^o_{sg}(X) = [+](\omega^1 + \omega^0) = [+](\omega + 1)$**.**

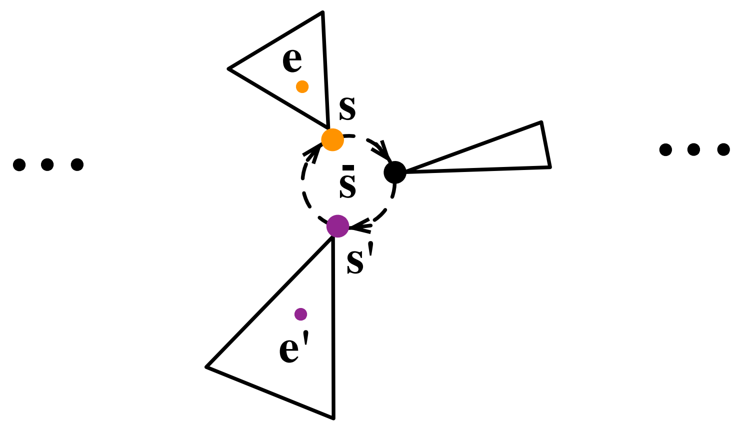


Concerning the tree-representation of finite semigroups

- **We clarify the organization of the tree-representation of finite semigroups.**
- **Some results about the kind of ω-subsets living in certain finite ω-semigroup follow.**

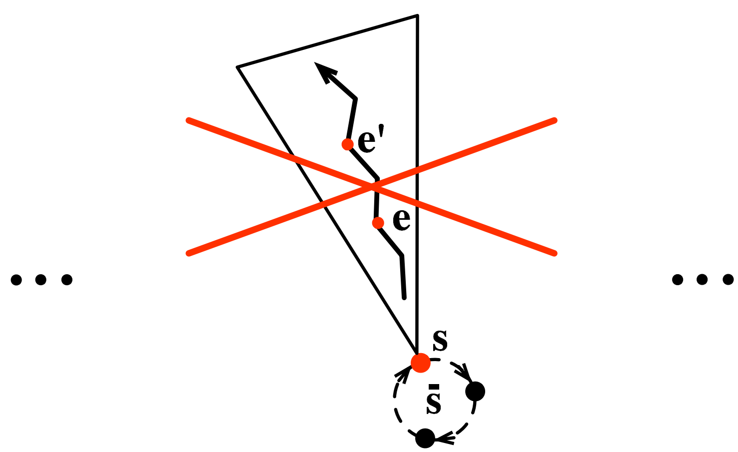
Lemma

Any two conjugate linked pairs $(s, e), (s', e')$ involve prefixes s, s' belonging to the same flower



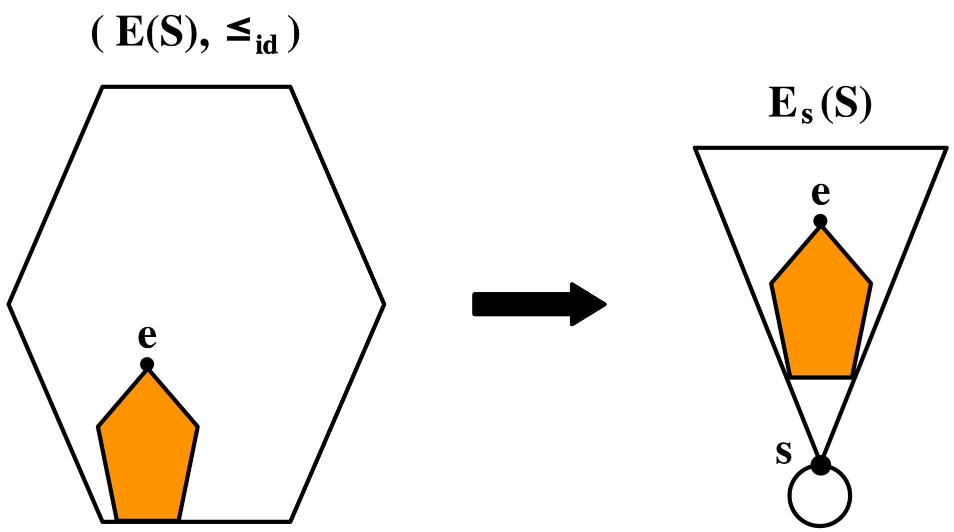
Lemma

Any two distinct conjugate idempotents cannot belong to the same chains. In particular, any two idempotents e, e' belonging to the conjugate linked pairs $(s, e), (s', e')$ cannot belong to the same chain.



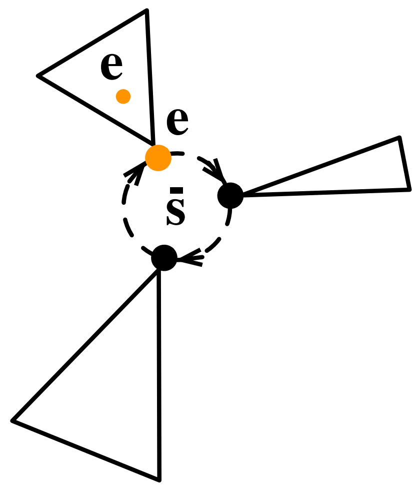
Lemma

Any idempotent appearing in a petal always carries with him the set of its $\leq_{id}$-predecessors



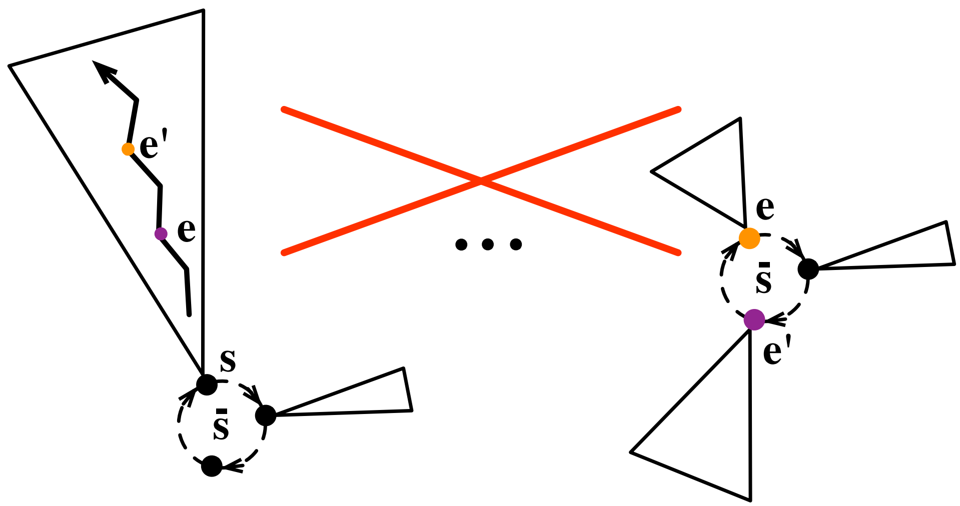
Lemma

Any idempotent is also a prefix somewhere

e idempotent ➡ 

Lemma

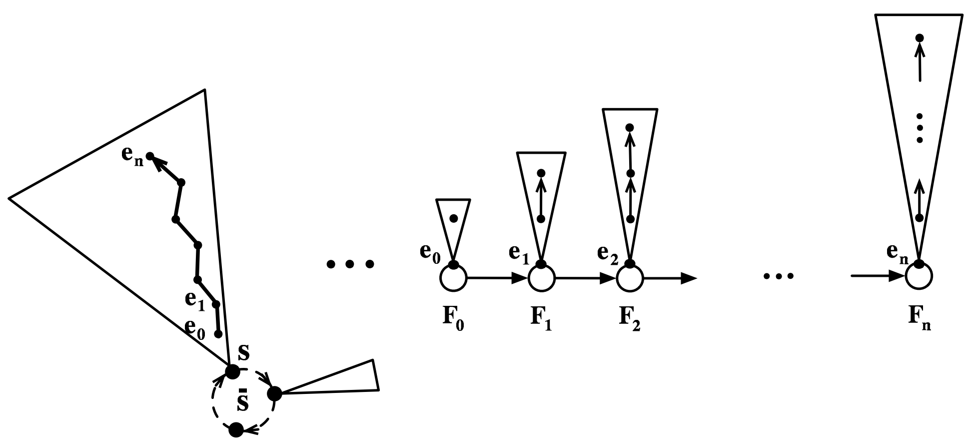
Any two idempotents of a same $\leq_{id}$-chain cannot be prefixes of a same flower.



Corollary

If there exists a $\leq_{id}$-chain b of length n in some petal, then:

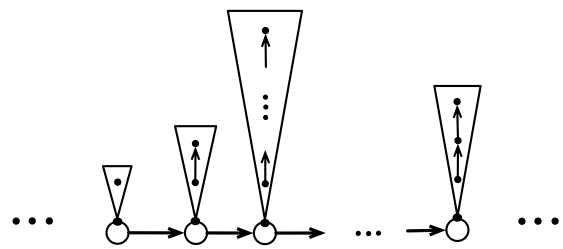
- **there exist at least $n+1$ flowers,**
- **these flowers are connected in a linear way such that the chain b is "growing" along the accessibility relation between flowers.**



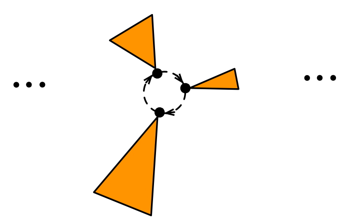
Remark

However, one has that

- **the petals may decrease along the $\leq_{pr}$-accessibility relation,**



- **a same flower may have petals of different heights.**



Proposition

There does not exist a family of ω-semigroups which exclusively characterizes ω-subsets of finite $\mathbb{SG}$-degrees.

Proposition

There does not exist a family of ω-semigroups which exclusively characterizes ω-subsets of $\mathbb{SG}$-degrees ω^n, for all $n > 0$.

Proposition

There does exist a family of ω-semigroups which exclusively characterize ω-subsets of $\mathbb{SG}$-degree $\omega^n \cdot p$, for all $n, p > 0$.

proof: consider finite ω-semigroups which provide a "linear" tree-representation.

Some Algebraic considerations

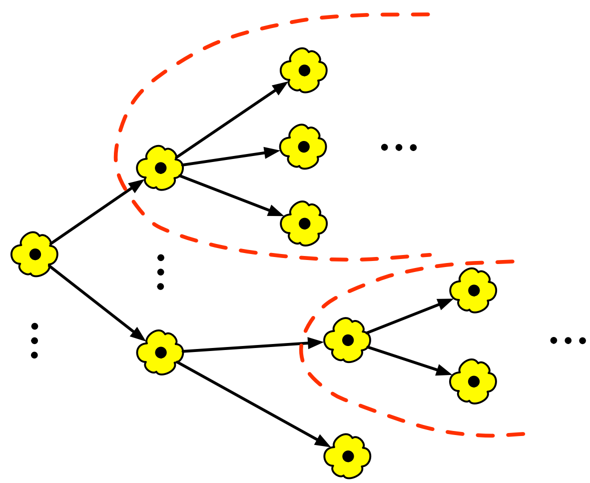
Remark

- (s, e) **is a linked pair iff both** e **is idempotent and is a right unit of** s**.**
- $s \leq_{pr} s'$ **iff** s **is a left divisor of** s'**.**
- $e \leq_{id} e'$ **iff** e **is a divisor of** e'**.**

Ideals

Lemma

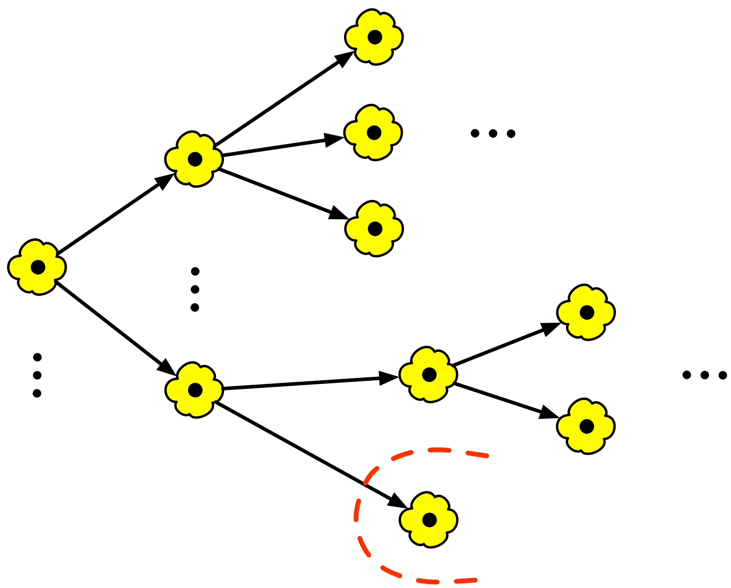
Any cut $\mathcal{C} = \bigcup_{i \in I} \bar{s}_i S$ of the tree-representation of S is a right ideal.



Notice that the converse is false...

Lemma

R is a minimal right ideal iff R is a terminal cut



Monoids

Proposition

Non-self-dual ω-subsets are exactly the ones living in finite ω-semigroups built on monoids, i.e.

Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and $X \subseteq S_\omega$, t.f.a.e.:

1. X **is non-self-dual**

2. $\exists\ M = (M_+, M_\omega)$ **s.t.** M_+ **is a monoid, and** $\exists\ Y \subseteq S_\omega$ **s.t.** $Y \equiv_{SG} X$

Left-cancelable semigroups

Definition

A semigroup S **is *left-cancelable* if** $\forall\ a, b, x \in S$**, one has** $xa = xb \ \Rightarrow \ a = b$**.**

Proposition

Let $S = (S_+, S_\omega)$ built on S_+ which is a finite left-cancelable semigroup, then $\forall\ X \subseteq S_\omega$, $d^O_{sg}(X) = 1$.

Groups

Corollary

Let $S = (S_+, S_\omega)$ **built on** S_+ **which is a finite group, then** $\forall\ X \subseteq S_\omega$, $d^o_{sg}(X) = 1$.

Cyclic semigroups

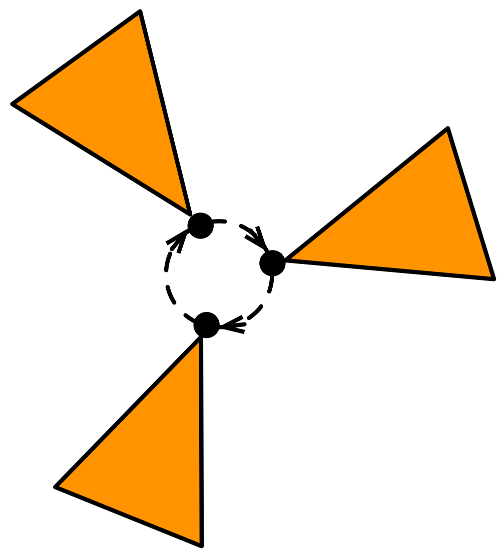
Proposition

Let $S = (S_+, S_\omega)$ **built on** S_+ **which is a finite cyclic semigroup, then** $\forall\ X \subseteq S_\omega$**,** $d^o_{sg}(X) = 1$**.**

Commutative semigroups

Lemma

The petals of a same flower are all identical.

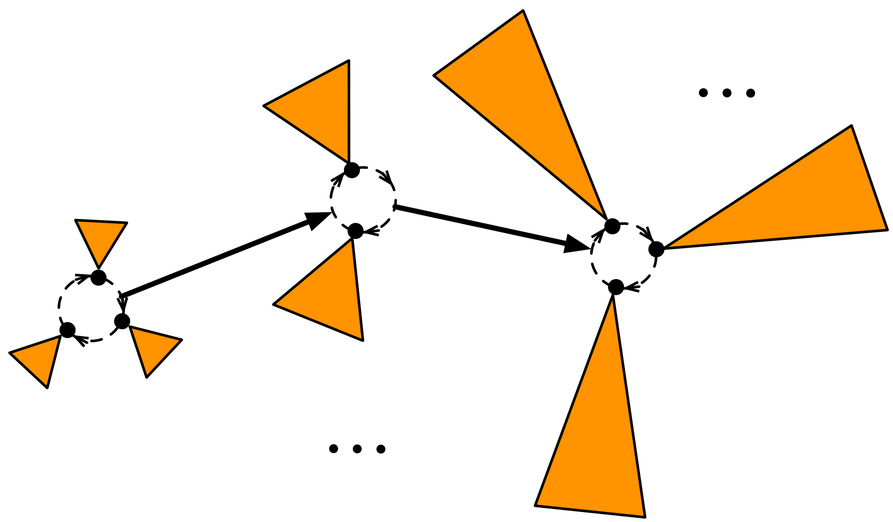


Lemma

One never has distinct conjugate linked pairs in a same petal.

Lemma

The petals are always increasing when going deeper in the tree-representation.



Lemma

The terminal flowers are all identical.

Corollary

Every commutative semigroup has a universal minimal right ideal.

Characterisic semigroups

Definition

Let S be a semigroup. The *characteristic semigroup* of S $(\Sigma(S), \circ)$ is defined by

1. $\Sigma(S) =_{def} \{U \subseteq S : U$ **is a subsemigroup of** $S\}$

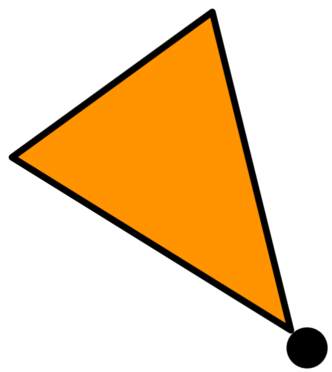
2. $B \circ B' =_{def} [B \cup B'] = \bigcup_{n \in \omega} (B \cup B')^n$

Remark

$\Sigma(S)$ is commutative and every element is idempotent.

Lemma

Any flower only contains a single petal.



Lemma

Let $(s, e), (s', e')$ **be two linked pairs, then**

$$(s, e) =_c (s', e') \;\Leftrightarrow\; (s, e) = (s', e').$$

Conclusions

- **Develop interactions between Algebra and Game Theory**
- **Try to find the algebraic correspondence of other machines, and characterize the hierarchy of their languages by this method.**