Introduction to Topology

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Outline

1. Metric spaces
2. Topological spaces
3. Compactness
4. Connectedness

Metric spaces

A *metric space* is a set of points and a distance which measures of the degree of closeness of pairs of points in this space.

Definition

A metric space is a pair (X, d), where X is a set and d is a distance on X, i.e. d is a function $d : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$,

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z)$.

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Example

1. $(\mathbb{R}, d)$ is a metric space, where d is defined by $d(a, b) = |b - a|$.
2. $(\mathbb{R}^n, d_n)$ is a metric space, where d_n is defined by $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

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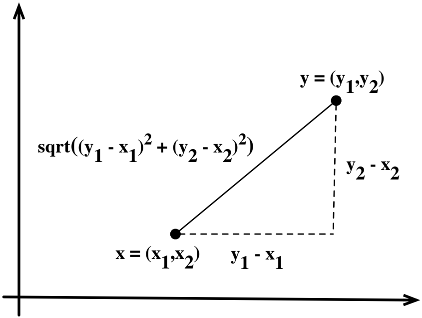
Example

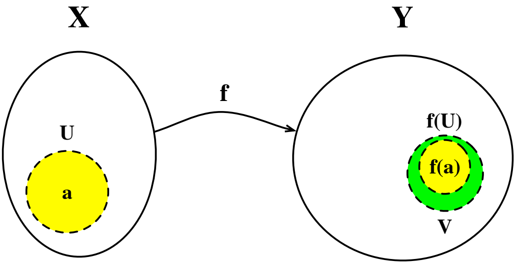
For $n = 2$, this distance is the classical distance used in analytical geometry:



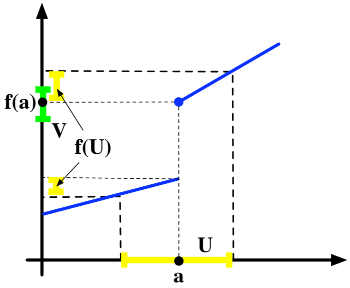
Example

Let A be an alphabet. Then (A^{ω}, d) is a metric space, with d is defined by $d(\alpha, \beta) = 2^{-r}$, where $r = \min\{n : \alpha(n) \neq \beta(n)\}$.

A function f from a metric space X to another metric space Y is *continuous at the point a* if for any (small) “neighborhood” of $f(a)$, denoted by V, there is a “neighborhood” of a, denoted by U, such that if $f(U) \subseteq V$.



The following function is not continuous at the point a.



By choosing such a V, there is no U such that $f(U) \subset V$.

Definition

Let (X, d) and (X', d') be two metric spaces, and let $a \in X$. A function $f : X \longrightarrow Y$ is continuous at the point a iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$.

A function $f : X \longrightarrow Y$ is continuous iff it is continuous at each point of X.

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$$\forall \varepsilon \; \exists \delta \; \forall x (P \Rightarrow Q).$$

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1. Let $f : (X, d) \longrightarrow (X', d')$ be a constant function, then f is continuous. Take any δ, say $\delta = 1$.
2. The identity function $id : (X, d) \longrightarrow (X, d)$ is continuous. Take $\delta = \varepsilon$.

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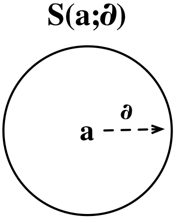
Definition

Let (X, d) be a metric space, and let $a \in X$ and δ be given. The open sphere about a of radius δ is the set

$$S(a; \delta) = \{x \in X : d(a, x) < \delta\}.$$



One has by definition

$$x \in S(a;\delta) \text{ iff } d(a,x) < \delta.$$

Therefore the definition of the continuity becomes...

Theorem

Let (X, d) and (X', d') be two metric spaces, and let $a \in X$. A function $f : X \longrightarrow Y$ is continuous at the point a iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that

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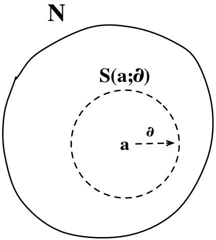
Definition

Let (X, d) and let $a \in X$. A subset N of X is called a neighborhood of a if there exists a $\delta > 0$ such that

$$S(a; \delta) \subset N.$$



Theorem

Let (X, d) and (X', d') be two metric spaces, and let $a \in X$. A function $f : X \longrightarrow Y$ is continuous at the point a iff for every neighborhood N of $f(a)$ there exists a corresponding neighborhood M of a such that

$$f(M) \subset N,$$

or equivalently

$$M \subset f^{-1}(N).$$

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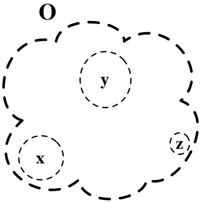
Proof.

On the blackboard... $\square$

Definition

A subset O of a metric space is said to be open if it is a neighborhood of each of its point.



Example

1. Every "open interval" $]a, b[$ is an open set of $\mathbb{R}$ (equipped with the usual metric).

$$]a, b[= \{x \in \mathbb{R} : a < x < b\},$$

2. Every cross product of "open intervals" $]a_i, b_i[$ is an open set of $\mathbb{R}^n$ (equipped with the usual metric).

$$\prod_{i=1}^{n}]a_i, b_i[.$$

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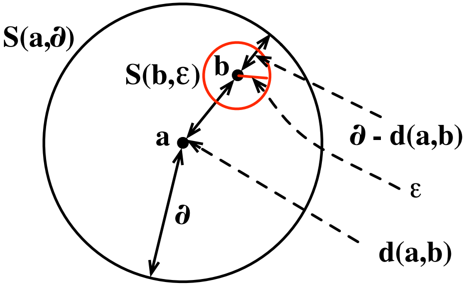
Lemma

Every open sphere is open.

Proof.

Let $S(a;\delta)$ be an open sphere. We need to prove that $S(a;\delta)$ is a neighborhood of each of its point. Let $b \in S(a;\delta)$ and let $\varepsilon < \delta - d(a,b)$. Then $S(b;\varepsilon) \subseteq S(a;\delta)$, which concludes the proof.



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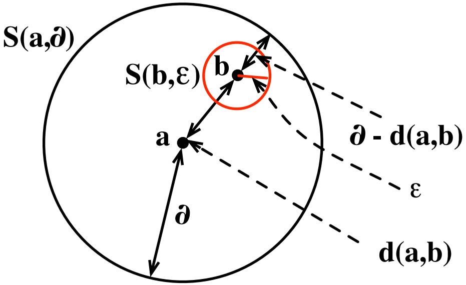

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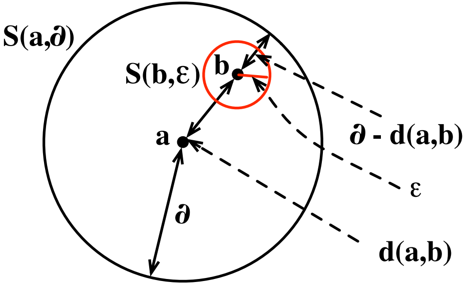

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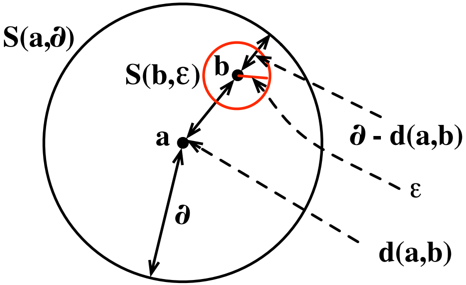

Theorem

Let (X, d) and (X', d') be two metric spaces and let $f : (X, d) \longrightarrow (X', d')$. Then f is continuous if and only if the inverse image of every open set of X' is an open set of X.

Proof.

On the blackboard. $\square$

Topological spaces

Definition

A *topological space* is a pair $(X, \mathcal{J})$, where X is a non-empty set and $\mathcal{J}$ is a collection of subsets of X such that

1. $X, \emptyset \in \mathcal{J}$,
2. If $O_1, \ldots, O_n \in \mathcal{J}$, then $O_1 \cap O_2 \cap \ldots \cap O_n \in \mathcal{J}$,
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Elements of the collection $\mathcal{J}$ are called the *open sets* of X.

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Example

Let $(X, \mathcal{J})$ be a topological space.

- $\emptyset$ is closed, since $\emptyset^c = X$ is open.
- X is closed, since $X^c = \emptyset$ is open.
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From metric spaces to topological spaces...

Let (X, d) be a metric space and let $\mathcal{T}$ be the collection of open sets of X, Then $(X, \mathcal{T})$ is a topological space.

From metric spaces to topological spaces...

Let (X, d) be a metric space and let $\mathcal{J}$ be the collection of open sets of X, Then $(X, \mathcal{J})$ is a topological space.

Example

1. The *real line* is the topological space arising from the metric space $(\mathbb{R}, d)$, where $d(a, b) = |b - a|$.
2. *The Euclidian n-space with the usual topology* is the topological space arising from the metric space $(\mathbb{R}^n, d)$.

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Example

1. Let X be a non-empty arbitrary set. Let $\mathcal{J} = \{\emptyset, X\}$. Then $(X, \mathcal{J})$ is a topological space.
2. Let X be a non-empty arbitrary set. Then $(X, \mathcal{P}(X))$ is a topological space. This topology on X is the one which contains the largest number of elements. It is called the *discrete topology*.

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Example

Let $X = \{a, b\}$. Let $\mathcal{J}_1 = \{\emptyset, X\}$, $\mathcal{J}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{J}_3 = \{\emptyset, \{b\}, X\}$, $\mathcal{J}_4 = \{\emptyset, \{a\}, \{b\}, X\}$. Then $(X, \mathcal{J}_i)$, for $i = 1, 2, 3, 4$, are four distinct topological spaces.

Example

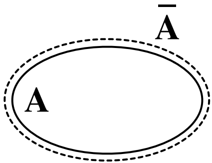
Let $\mathbb{N}$ be the set of positive integers. For each $n \in \mathbb{N}$, let $O_n = \{n, n+1, n+2, \ldots\}$, and let $\mathcal{J} = \{\emptyset, O_0, O_1, O_2, \ldots\}$. Then $(\mathbb{N}, \mathcal{J})$ is a topological space.

Definition

Let $(X, \mathcal{J})$ and $(X', \mathcal{J}')$ be two topological spaces and let $f : X \longrightarrow Y$. Then f is said to be *continuous* if and only if the preimage of any open set is an open set.

Definition

The *closure* $\bar{A}$ of a subset A is the smallest closed set containing A.



Lemma

A is closed if and only if $A = \bar{A}$.

Proof.

($\Rightarrow$) Suppose that A is closed. Then the smallest closed set containing A is A itself, i.e $\bar{A} = A$.

($\Leftarrow$) Suppose $A = \bar{A}$. Since $\bar{A}$ is closed by definition, then A is also closed.

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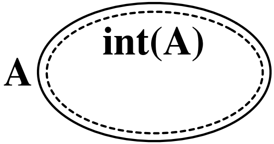
Definition

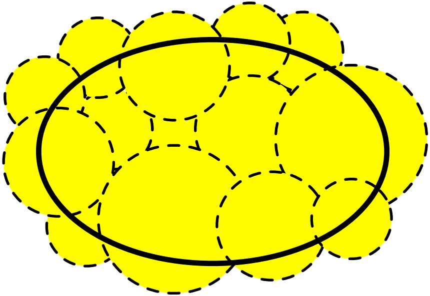
The *interior* $int(A)$ of a subset A is the largest open set contained in A.



Compactness

A *compact* set is a closed and bounded set.

Bounded in the sense that it can be covered with a finite number of open sets.



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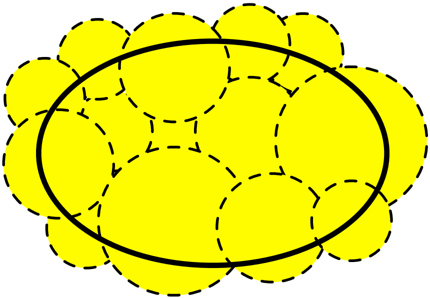

Compact set

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Let X be a set and let $(A_i)_{i\in\alpha}$ be a family of sets. The collection $(A_i)_{i\in\alpha}$ is called a *covering* of X if $X \subseteq \bigcup_{i\in\alpha} A_i$.

Given $\beta \subseteq \alpha$, if the the subfamily $(A_i)_{i\in\beta}$ still covers X, then $(A_i)_{i\in\beta}$ is called a *subcovering* of $(A_i)_{i\in\alpha}$.

If A_i is open for each $i \in \alpha$, then $(A_i)_{i\in\alpha}$ is called an *open covering* of X.

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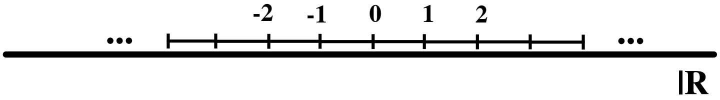
Example

For each integer n, let $A_n = [n, n+1]$. Then $(A_n)_{n \in \mathbb{Z}}$ is an infinite covering of $\mathbb{R}$.



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Lemma

The closed interval $[0, 1]$ *is compact.*

Towards a contradiction, suppose that $[0,1]$ is not compact. Then there exists an open covering $(O_\alpha)_{\alpha \in I}$ of $[0,1]$ from which one cannot extract a finite subcovering of $[0,1]$.

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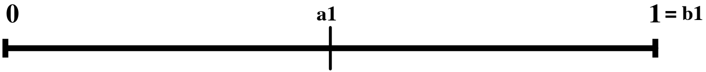

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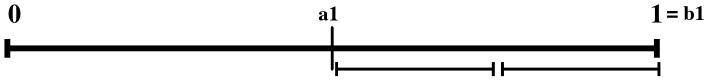

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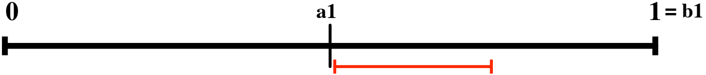

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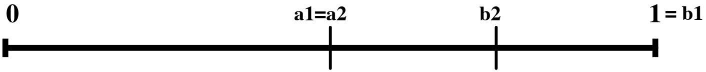

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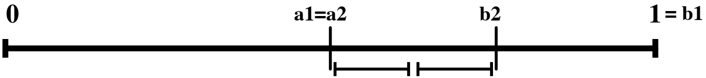

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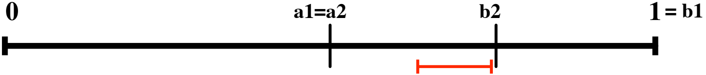

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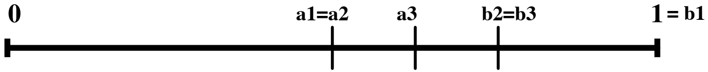

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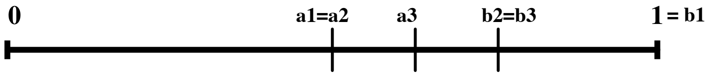
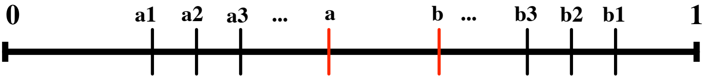


And so on, and so forth.

For each $n > 0$, one has

- $|b_n - a_n| = \frac{1}{2^n}$,
- $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. (1)

Now let a be the smallest upper bound of the $(a_i)_{i>0}$, and let b be the greatest lower bound of the $(b_i)_{i>0}$



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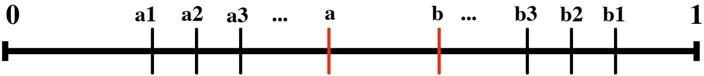

Then (1) implies $a_n \leq a \leq b \leq b_n$, for all $n > 0$.

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Since the covering $(O_\alpha)_{\alpha \in I}$ covers $[0,1]$ and $a = b \in [0,1]$, there exists an open set O_β such that $a \in O_\beta$.

Since O_β is open, there is an $\varepsilon > 0$ such that $S(a;\varepsilon) \subseteq O_\beta$.

Now let N large enough such that $\frac{1}{2^N} < \varepsilon$. Then $b_N - a_N = \frac{1}{2^N} < \varepsilon$.

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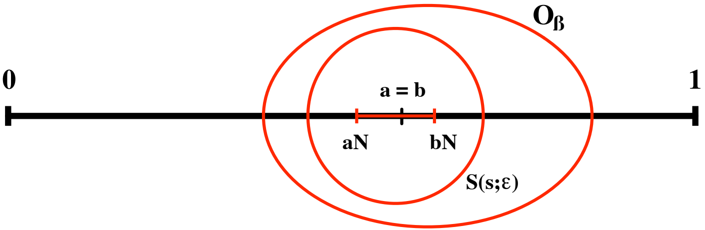

In other terms, $[a_N, b_N]$ is covered with a finite subcovering of $(O_\alpha)_{\alpha\in I}$, namely O_β itself. A contradiction.

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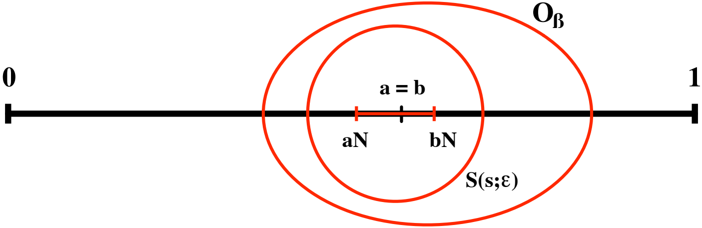

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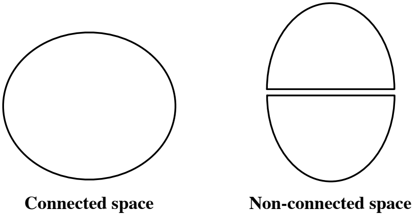
Corollary

Each closed interval $[a, b]$ *is compact.*

Connectedness

A topological space is *connected* if it is all of one piece. That is if it is impossible to decompose it into two disjoint non-empty open sets.



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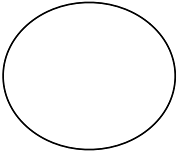
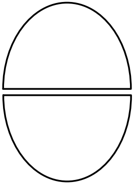


Connected space



Non-connected space

Definition

A topological space X is said to be *connected* if the only two clopen sets of X are $\emptyset$ and X itself.

Lemma

A topological space X is not connected *if and only if there are two non-empty open sets P and Q such that*

- $P \cup Q = X$,
- $P \cap Q = \emptyset$.

Proof.

($\Rightarrow$) If X is not connected, there exists a clopen set C such that $C \neq \emptyset, X$. Therefore C and C^c are open sets, and $C \cup C^c = X$ and $C \cap C^c = \emptyset$.

($\Leftarrow$) If there are two non-empty open sets P and Q such that $P \cup Q = X$ and $P \cap Q = \emptyset$, then obviously $P = Q^c$ and $Q = P^c$. These properties imply P and Q clopen different from $\emptyset$ and X.

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$\square$

One can prove that connected subsets of the real line $\mathbb{R}$ are exactly the intervals.