Automata, Semigroups and Games

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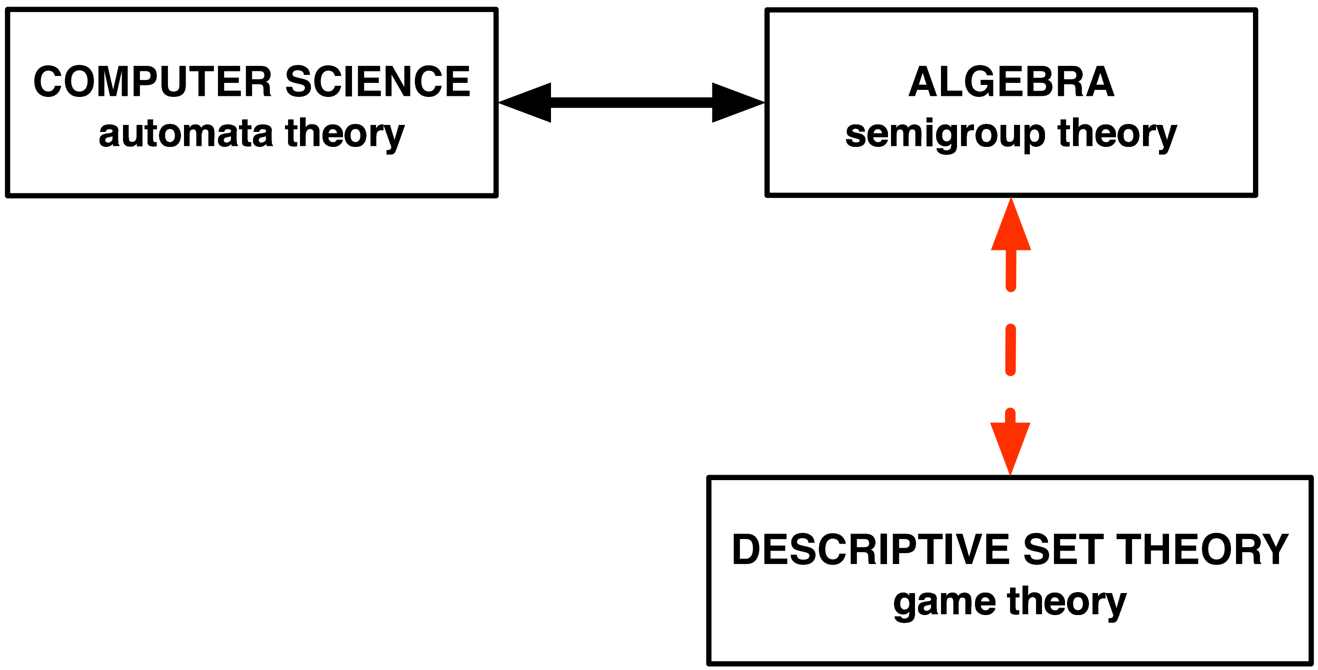

COMPUTER SCIENCE
automata theory
ALGEBRA
semigroup theory
DESCRIPTIVE SET THEORY
game theory

COMP. SCIENCE		ALGEBRA
automaton **(rational language)**	$\longleftrightarrow$	**finite semigroup**
Büchi automaton **(ω-rational language)**	$\longleftrightarrow$	**Wilke algebra** **finite $\boldsymbol{\omega}$-semigroup**

ω-*semigroup* $S = (S_+, S_\omega)$

- $(S_+, \cdot)$ **is a semigroup,** S_ω **is a set**
- $\pi : {S_+}^\omega \longrightarrow S_\omega$ **an infinite product** ω**-associative**

$$\pi(s_0, s_1, s_2, \ldots) = \pi(s_0 \cdot s_1 \cdot \ldots \cdot s_{n_0}, s_{n_0+1} \cdot \ldots \cdot s_{n_1}, \ldots)$$

free ω-*semigroup*

$$S = (S_+, S_\omega = S_+^\omega)$$

where the infinite product is the identity

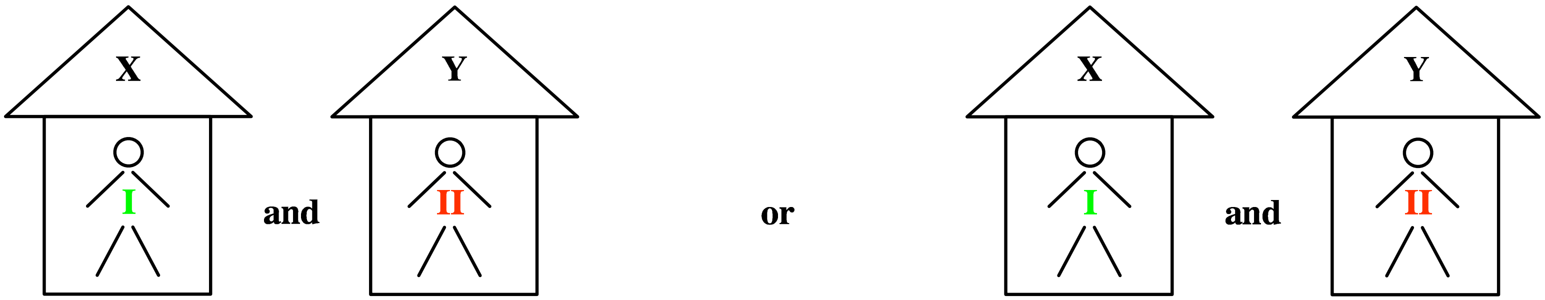
An infinite two players game $\boldsymbol{SG(X,Y)}$ on ω-semigroups

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-semigroups and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$**.**

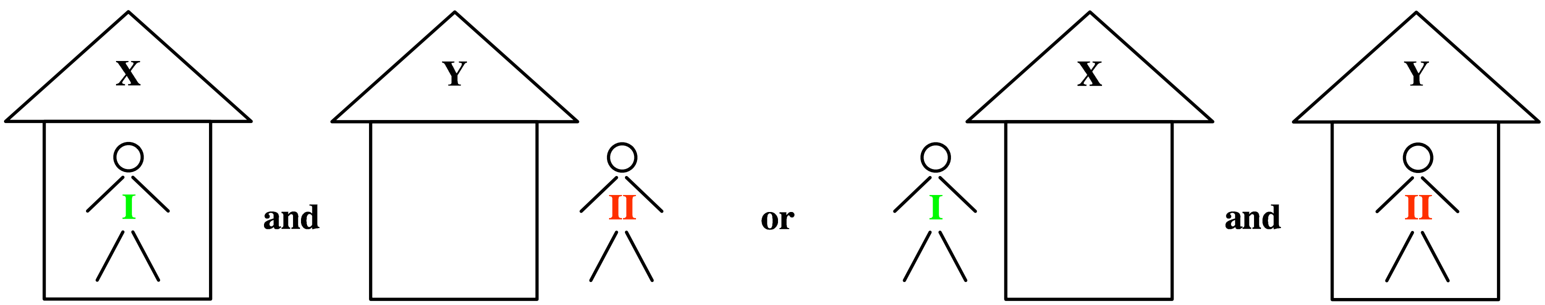
(X) **I**	s_0	s_1	$\ldots$		$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle s_0, s_1, s_2, \ldots \rangle$
(Y) **II**	t_0	t_1		$\ldots$	$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle t_0, t_1, t_2, \ldots \rangle$

II wins $\Leftrightarrow_{def} \big(\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y\big)$

II wins in $SG(X,Y)$ iff



I wins in $SG(X,Y)$ iff



- **by Borel determinacy:**

  game $SG(X, Y)$ is determined for X and Y Borel subsets of S_ω and T_ω

- **$\rightarrow$ topology on S_ω:**

  – **a bad idea:**

  $$X \subseteq S_\omega \textbf{ is open } \Leftrightarrow_{def} X = s \cdot S_\omega$$

  when S_+ is a group, only two Borel sets are $\{\emptyset, S_\omega\}$

  – **a good idea:**

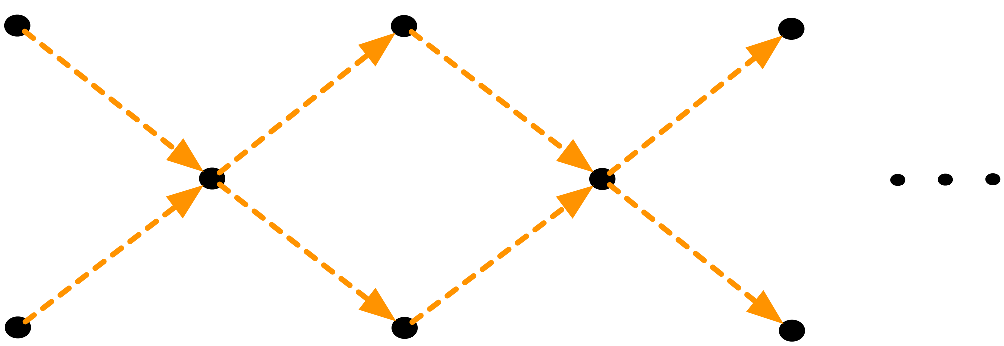
  $$X \subseteq S_\omega \textbf{ is open } \Leftrightarrow_{def} \pi_S^{-1}(X) \textbf{ open of } S_+^\omega$$

  S_+^ω equipped with the product top. of the discrete top. on S_+

***SG*-ordering**

$$\boldsymbol{X} \leq_{\boldsymbol{SG}} \boldsymbol{Y} \Leftrightarrow_{def} \textbf{II has a w. s. in } SG(X,Y)$$

- **partial ordering**
- **well founded**
- **antichains of length at most 2**



Properties of the SG-hierarchy

- **restricted to free ω-semigroups:**

$$SG\textbf{-hierarchy} \equiv \textbf{Wadge hierarchy}$$

- **restricted to finite ω-semigroups:**

$$SG\textbf{-hierarchy} \equiv \textbf{Wagner hierarchy}$$

let $S = (S_+, S_\omega)$ **be an** ω**-semigroup and** $X \subseteq S_\omega$

$$X \not\leq_{SG} X^C$$

$$\Leftrightarrow$$

S_+ **"is" a monoid**

$$\Leftrightarrow$$

player in charge of X **is allowed to skip**

let $S = (S_+, S_\omega)$ **be an** ω**-semigroup and** $X \subseteq S_\omega$

$$\boxed{\begin{array}{c} X \equiv_{SG} s \cdot X, \ \forall s \in S_+ \\ \Leftrightarrow \\ S_+ \text{ ''is'' a group} \\ \Leftrightarrow \\ \text{player in charge of } X \text{ is allowed to erase} \end{array}}$$

- **important hierarchies are particular cases of SG-hierarchy**
- **essential algebraic notions expressed in a very natural game theoretical way**
- **work in progress...**