The Algebraic Counterpart of the Wagner Hierarchy

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Outline

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Classifying ω-rational languages

Automata
The Wagner Hierarchy
[Wag79]

Descriptive set theory
The Wadge hierarchy
of ω-rational languages
[Sel98]

Algebra
[CarPer96,99], [DupRis06]
The FSG-hierarchy

The Wagner Hierarchy
(most refined classification of ω-rational languages)

Given two ω-rational languages K and L, we set

$$K \leq_W L \quad \text{iff} \quad K = f^{-1}(L) \text{ for some continuous function } f$$
$$\text{iff} \quad w \in K \Leftrightarrow f(w) \in L \text{ for some continuous function } f$$
$$\text{iff} \quad \text{Player II has a w.s. in the Wadge game } W(K,L)$$

Then as usual, $K <_W L$ iff $K \leq_W L \not\leq_W K$ and $K \equiv_W L$ iff $K \leq_W L \leq_W K$.

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The collection of all ω-rational languages ordered by the relation $\leq_W$ is called *the Wagner hierarchy*.

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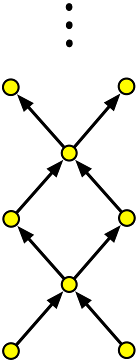

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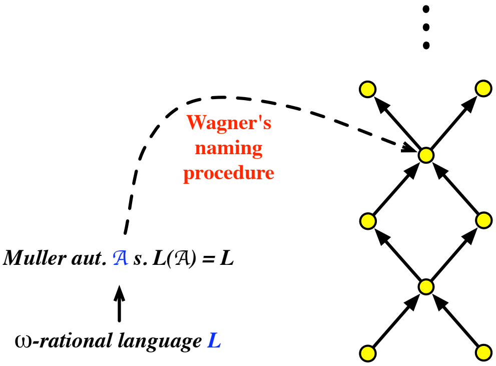

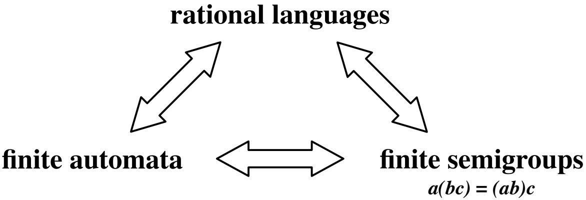
rational languages
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ω-rational languages

finite ω-automata

finite ω-semigroups

$a(bc) = (ab)c$ & infinite product

Definition (ω-semigroup)

An ω-semigroup S is a pair (S_+, S_ω), where S_+ is a semigroup, S_ω is a set, and equipped with

- an associative mixed product : $S_+ \times S_\omega \longrightarrow S_\omega$
- a surjective infinite product $\pi_S : S_+^\omega \longrightarrow S_\omega$ satisfying

$$\pi_S(s_0 s_1 \cdots s_{k_1 - 1}, s_{k_1} \cdots s_{k_2 - 1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$
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Finite pointed ω-semigroups are the algebraic counterparts of Büchi automata.

Theorem

An ω-language is recognizable by a finite pointed ω-semigroup iff it is recognizable by a finite Büchi (or Muller) automaton (iff it is ω-rational).

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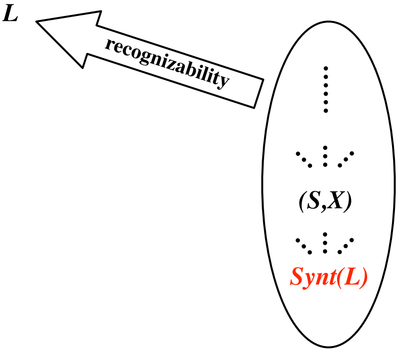
Among all finite pointed ω-semigroups recognizing a given ω-rational language L, there exists a minimal one, **the syntactic ω-semigroup** of L, denoted by $Synt(L)$.



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Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two finite ω-semigroups, and let also $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.

The infinite two-player game $\mathbb{SG}(X, Y)$ is defined as follows:

- Player I plays elements from S_+,
- Player II plays elements from T_+,
- players I and II play alternately, Player I begins,
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& & \searrow & & \nearrow & & \searrow & & & & \\
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- Player II wins the game $\mathbb{SG}(X, Y)$ iff

$$\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y.$$

Definition ($\mathbb{SG}$-reduction)

We write $(S, X) \leq_{SG} (T, Y)$ iff Player II has a winning strategy in $\mathbb{SG}(X, Y)$. And as usual $(S, X) \equiv_{SG} (T, Y)$ iff $(S, X) \leq_{SG} (T, Y) \leq_{SG} (S, X)$.

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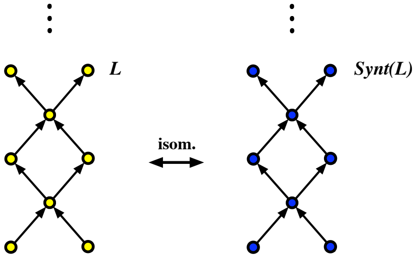

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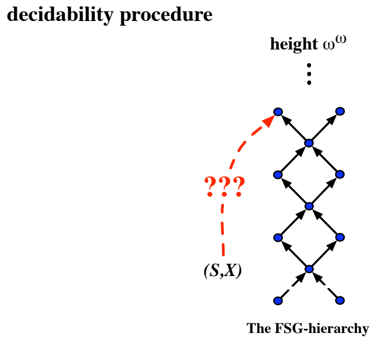

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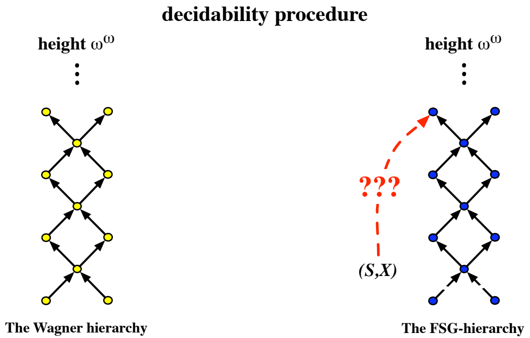

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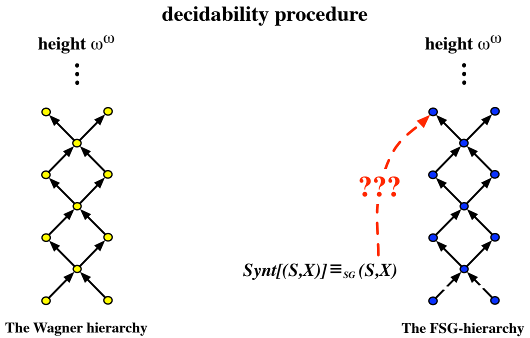

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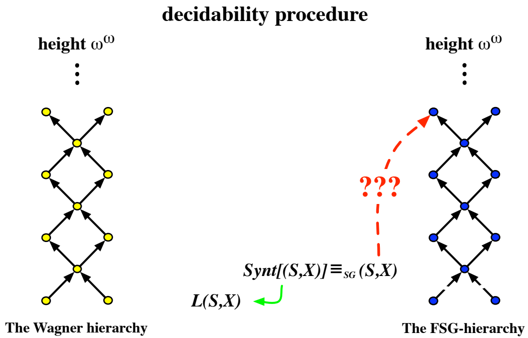

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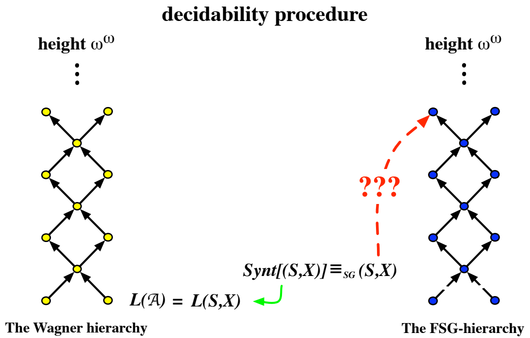

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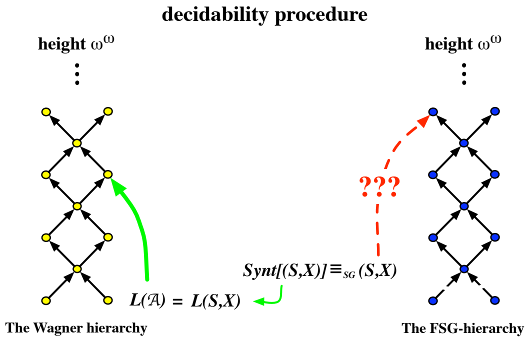

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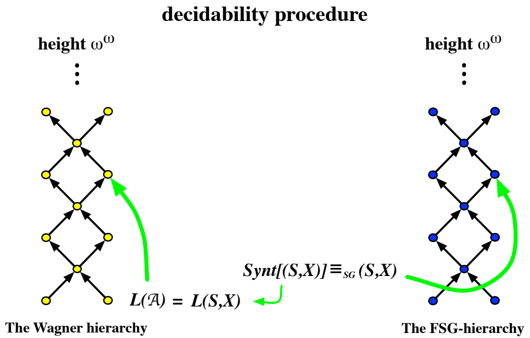

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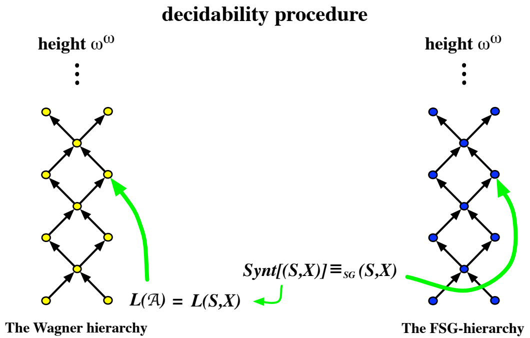

Corollary

Let L be an ω-rational language. Then $d_W(L) = \alpha$ iff $d_{SG}(Synt(L)) = \alpha$.

Conclusions

- The $\mathbb{FSG}$-hierarchy is the algebraic counterpart of the Wagner hierarchy.
- Decidability procedure of the $\mathbb{FSG}$-hierarchy.
- **One can compute the Wagner degree of an ω-rational language directly on its syntactic image.**
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