The Algebraic Counterpart of the Wagner Hierarchy

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Outline

1 Introduction

2 ω-Rational Languages

3 The Wagner hierarchy

4 ω-Semigroups

5 The $\mathbb{FSG}$-hierarchy

6 Conclusion

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Classifying ω-rational languages

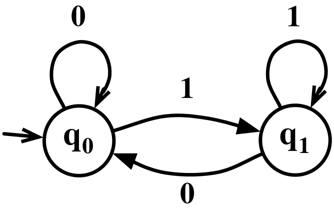
Automata
The Wagner Hierarchy
[Wag79]

Descriptive set theory
The Wadge hierarchy
of ω-rational languages
[Sel98]

Algebra
[CarPer96,99], [DupRis06]
The FSG-hierarchy

Muller automaton (deterministic)

$\mathcal{A}$ is a labeled graph, with $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{P}^{Q}$.



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Then $L^{\omega}(\mathcal{A}) = A^{*}(0^{\omega} \cup 1^{\omega})$.

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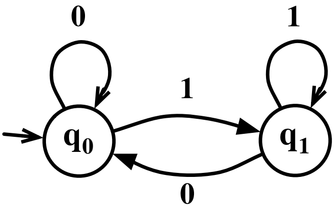

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Theorem

The following conditions are equivalent:

1. L *is* ω*-rational*
2. L *is recognizable by a finite Muller automaton.*

The Wagner Hierarchy
(classification of ω-rational languages)

Given two ω-rational languages K and L, we set

$$\begin{aligned} K \leq_W L \quad &\text{iff} \quad K = f^{-1}(L) \text{ for some continuous function } f \\ &\text{iff} \quad w \in K \Leftrightarrow f(w) \in L. \\ K <_W L \quad &\text{iff} \quad K \leq_W L \not\leq_W K \\ K \equiv_W L \quad &\text{iff} \quad K \leq_W L \leq_W K. \end{aligned}$$

Definition (Wagner hierarchy)

The collection of all ω-rational languages ordered by the relation $\leq_W$ is called *the Wagner hierarchy*.

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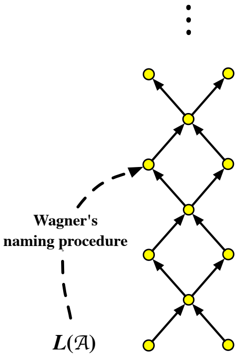
Theorem

The Wagner hierarchy has height ω^ω and it is decidable.



Wagner's decidability procedure

Classifying ω-rational language is equivalent to classifying their underlying Muller automata.

$/$

$$\mathcal{T} = \{\{q_0\}, \{q_2, q_3\}, \{q_4\}, \{q_6, q_7\}, \{q_8\}, \{q_{10}, q_{11}\}, \{q_{12}\}\}$$

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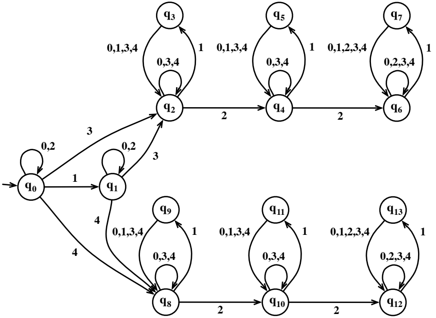

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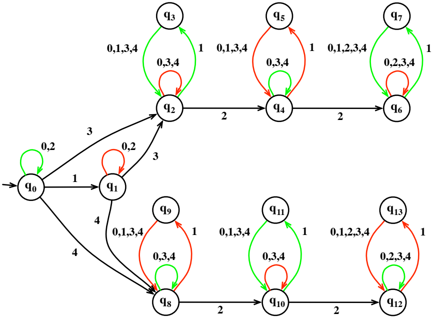

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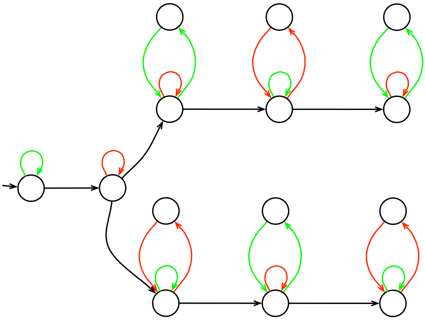

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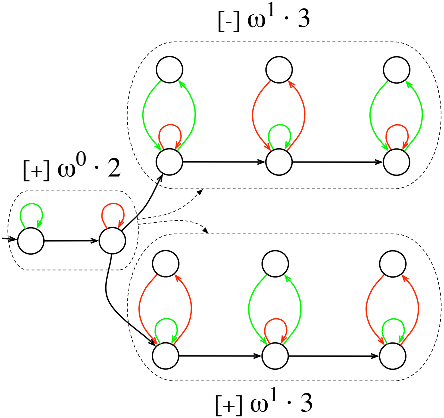

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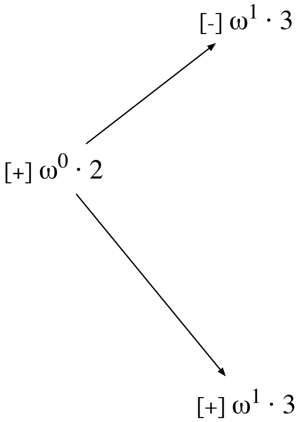

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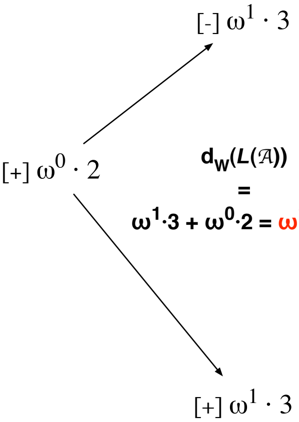

$$\mathbf{d_W(L(\mathcal{A}))} = \omega^1 \cdot 3 + \omega^0 \cdot 2 = \omega^1 \cdot 3 + 2$$

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rational languages

finite automata **finite semigroups**

$a(bc) = (ab)c$

ω-rational languages

finite ω-automata

finite ω-semigroups

$a(bc) = (ab)c$ **& infinite product**

A semigroup $(S, \cdot)$ is a set S equipped with an associative binary operation "$\cdot$" on S.

Example

The set $\{0, 1\}$ equipped with the usual multiplication is a semigroup.

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An ω-semigroup is a semigroup equipped with a suitable infinite product.

Definition (ω-semigroup)

An ω-semigroup S is a pair (S_+, S_ω), where S_+ is a semigroup, S_ω is a set, and equipped with

- an associative mixed product : $S_+ \times S_\omega \longrightarrow S_\omega$
- a surjective infinite product $\pi_S : S_+^\omega \longrightarrow S_\omega$ satisfying

$$\pi_S(s_0 s_1 \cdots s_{k_1-1}, s_{k_1} \cdots s_{k_2-1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$

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Notation: $\pi_S(x, x, x, x, \ldots)$ will be written x^ω.

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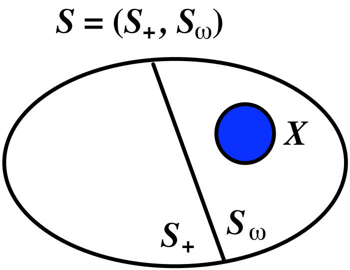
Example

$S = (\{0,1\}, \{a\})$ is an ω-semigroup for the following operations:

- the set $\{0,1\}$ is equipped with the usual multiplication,
- every infinite product is equal to a.

A *pointed ω-semigroup* is a pair (S, X), where

- $S = (S_+, S_\omega)$ is an ω-semigroup,
- $X \subseteq S_\omega$.



Finite pointed ω-semigroups are the algebraic counterparts of Büchi automata.

Theorem

An ω-language is recognizable by a finite pointed ω-semigroup iff it is recognizable by a finite Büchi (or Muller) automaton (iff it is ω-rational).

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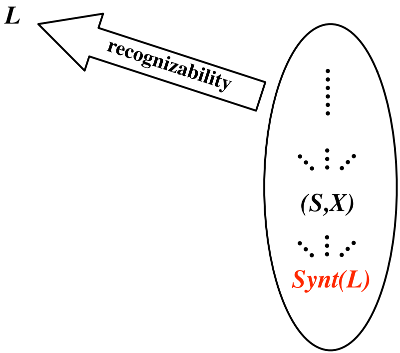
Among all finite pointed ω-semigroups recognizing a given ω-rational language L, there exists a minimal one, **the syntactic ω-semigroup** of L, denoted by $\boldsymbol{Synt(L)}$.



Example

Consider the language $K = ((a+b)^*a)^\omega$. Then $Synt(K) = (S, X)$, where

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 0 \qquad 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1$$
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- $X = \{0^\omega\}$.

Example

Consider the language $L = (a\{b,c\}^* \cup \{b\})^\omega$. Then $Synt(L) = (T, Y)$, where

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

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a^2 = a & ab = a & ac = a & ba = a \\
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We aim to classify finite pointed ω-semigroups. We adopt a hierarchical game approach.

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2. We define the corresponding reduction relation.
3. We then describe the resulting hierarchy.

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Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let also $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.

The infinite two-player game $\mathbb{SG}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as follows:

- Player I plays elements from S_+,
- Player II plays elements from T_+,
- players I and II play alternately, Player I begins,
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\begin{array}{llllllll}
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& & \searrow & \nearrow & & \searrow & & & \\
\mathrm{II}: & & - & & & t_0 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (t_0, t_1, t_2, \cdots)
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- Player II wins the game $\mathbb{SG}(X, Y)$ iff

$$\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y.$$

Definition ($\mathbb{SG}$-reduction)

We write $X \leq_{SG} Y$ iff Player II has a winning strategy in $\mathbb{SG}(X, Y)$. And as usual $X \equiv_{SG} Y$ iff $X \leq_{SG} Y \leq_{SG} X$.

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Example

Let $S = (\{0, 1\}, \{0^{\rightarrow}0, 0^{\rightarrow}1, 1^{\rightarrow}1\})$ be the ω-semigroup defined by the operations:

$0 \cdot 0 = 0$ $\quad 0 \cdot 1 = 0$ $\quad 1 \cdot 0 = 0$ $\quad 1 \cdot 1 = 1$

$00^{\omega} = 0^{\rightarrow}0$ $\quad 10^{\omega} = 0^{\rightarrow}0$ $\quad 01^{\omega} = 0^{\rightarrow}1$ $\quad 11^{\omega} = 1^{\rightarrow}1$

- $\pi_S(\alpha) = 0^{\rightarrow}0$ iff α contains infinitely many 0's,
- $\pi_S(\alpha) = 0^{\rightarrow}1$ iff α contains finitely many 0's,
- $\pi_S(\alpha) = 1^{\rightarrow}1$ iff $\alpha = 1^{\omega}$.

Then $\{1^{\rightarrow}1\} \leq_{SG} \{0^{\rightarrow}0\}$.

Example

Let $S = (\{0,1\}, \{0^{\to}0, 0^{\to}1, 1^{\to}1\})$ be the ω-semigroup defined by the operations:

$0 \cdot 0 = 0$ $\quad$ $0 \cdot 1 = 0$ $\quad$ $1 \cdot 0 = 0$ $\quad$ $1 \cdot 1 = 1$

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Example (continued)

We give a w. s. for Player II in $\mathbb{SG}\left(\{1^{\rightarrow}1\},\{0^{\rightarrow}0\}\right)$:

Player I	1	1	1	$\cdots\ \cdots$	$1^{\rightarrow}1 \in \{1^{\rightarrow}1\}$
Player II	0	0	0	$\cdots\ \cdots$	$0^{\rightarrow}0 \in \{0^{\rightarrow}0\}$

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Definition ($\mathbb{FSG}$-hierarchy)

The collection of finite pointed ω-semigroups ordered by the $\leq_{SG}$-relation is called *the $\mathbb{FSG}$-hierarchy*.

Proposition ($\mathbb{SG}$-Borel Determinacy)

Let (S, X) and (T, Y) be two finite pointed ω-semigroups. Then the game $\mathbb{SG}(X, Y)$ is determined.

Proof.

A consequence of Borel Wadge determinacy. A given player has a w. s. in $\mathbb{SG}(X, Y)$ iff this same player has a w. s. in $\mathbb{W}(\pi_S^{-1}(X), \pi_T^{-1}(Y))$. $\square$

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- *The $\leq_{SG}$-antichains have length at most two.*
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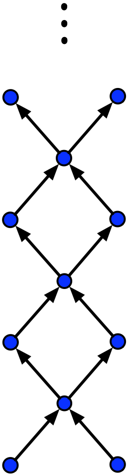
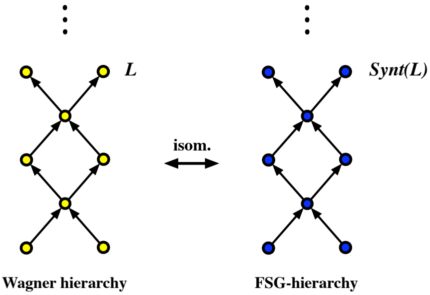
The $\mathbb{FSG}$-hierarchy



Theorem

The $\mathbb{FSG}$-hierarchy and the Wagner hierarchy are isomorphic.



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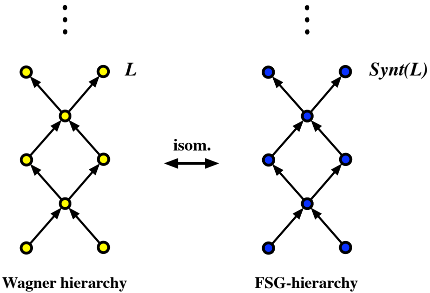

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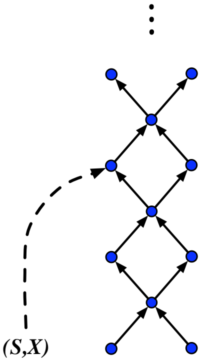
Completely algebraic decidability procedure: **one can compute the Wagner degree of an ω-rational language directly on its syntactic image.**



Example

Consider the syntactic pointed ω-semigroup of $K = ((a+b)^*a)^\omega$:

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0\cdot 0 = 0 \quad 0\cdot 1 = 0 \quad 1\cdot 0 = 0 \quad 1\cdot 1 = 1$$
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Example (continued)

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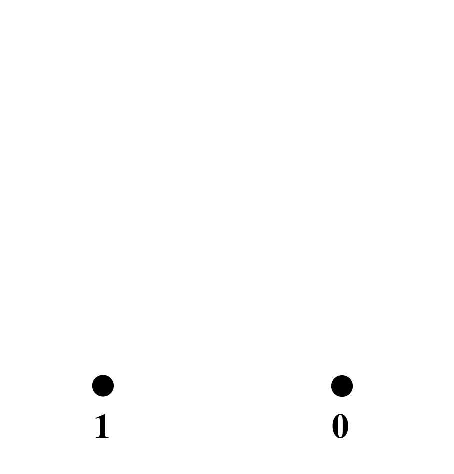

Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example (continued)

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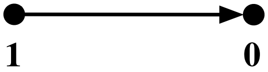

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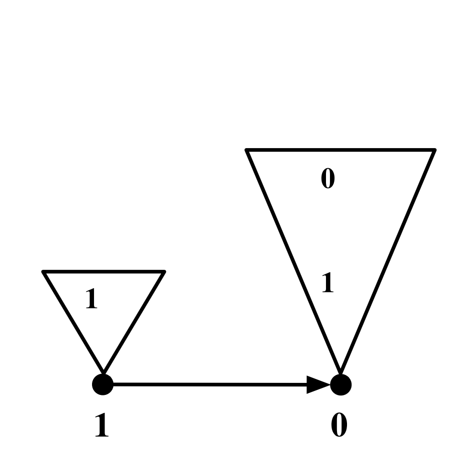

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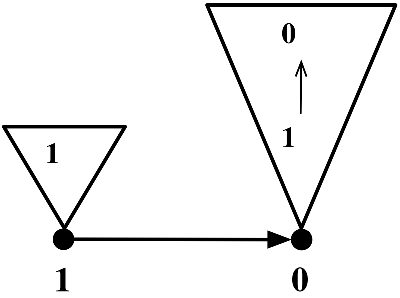

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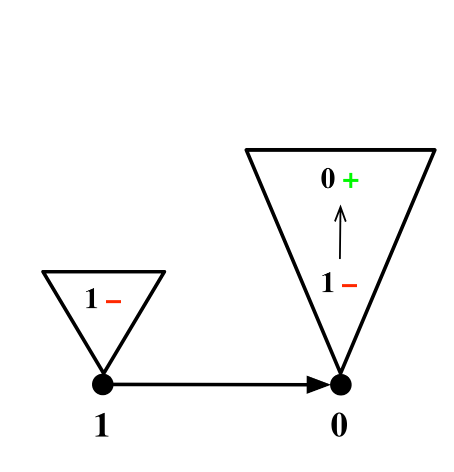

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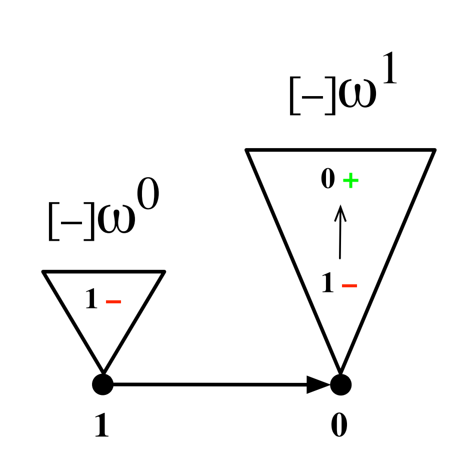

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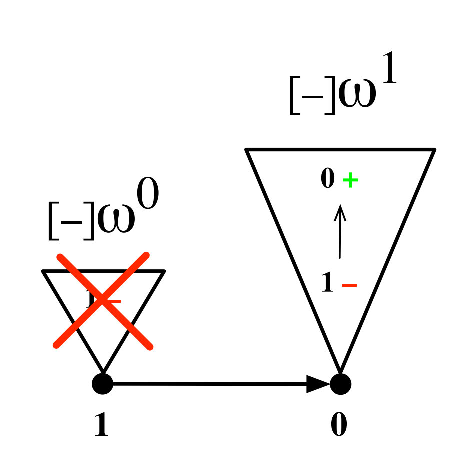

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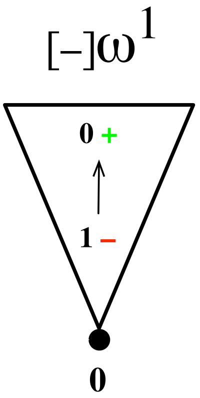

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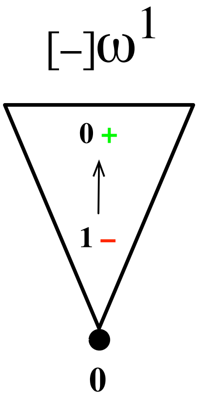

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Example

Consider the syntactic pointed ω-semigroup of $L = (a\{b,c\}^* \cup \{b\})^\omega$:

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

$$
\begin{array}{cccc}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega
\end{array}
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- $Y = \{a^\omega\}$.

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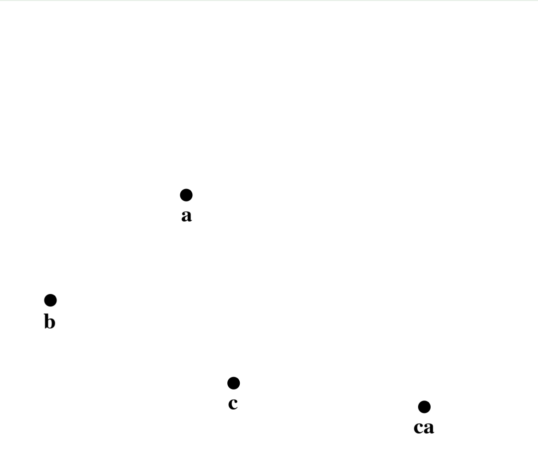

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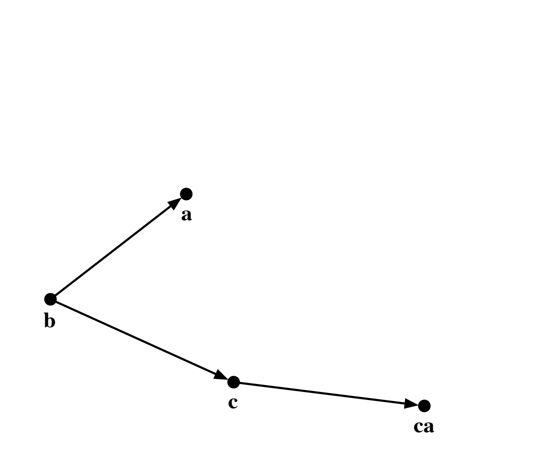

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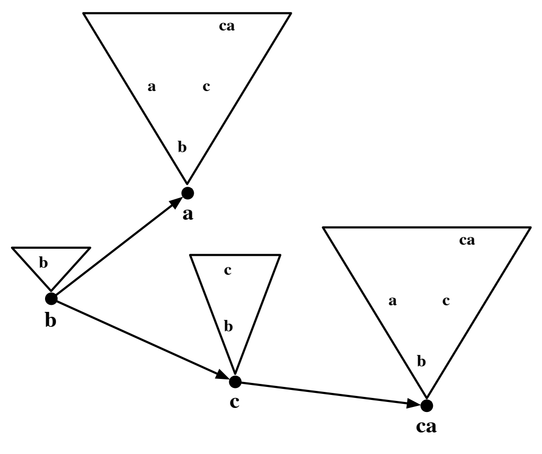

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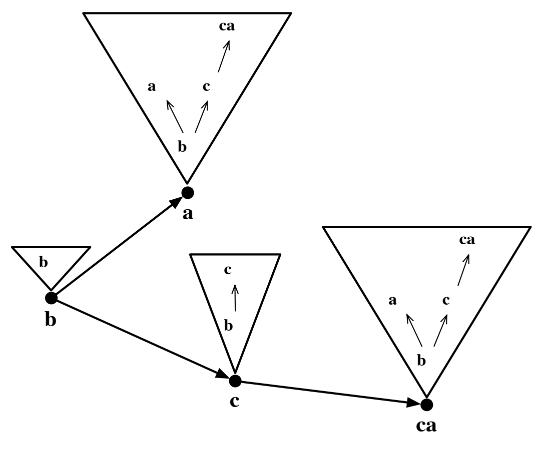

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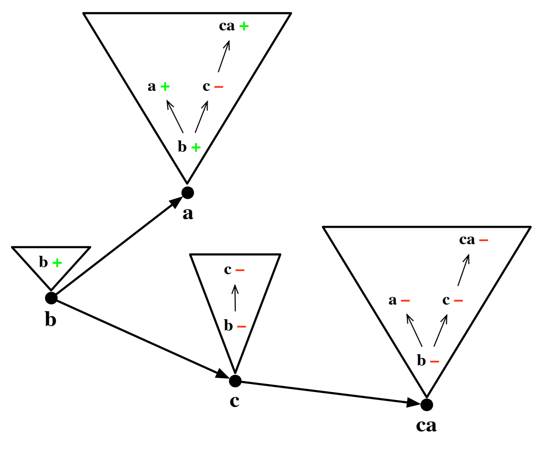

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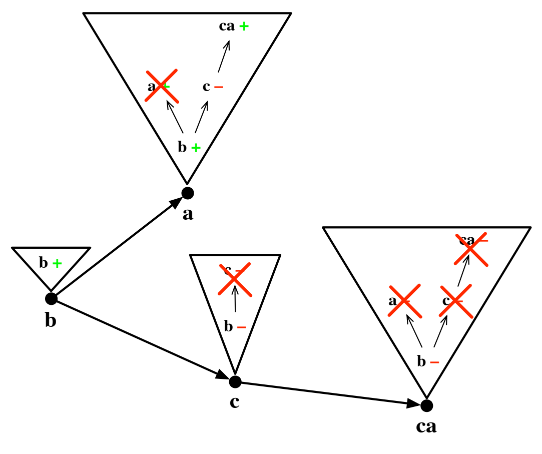

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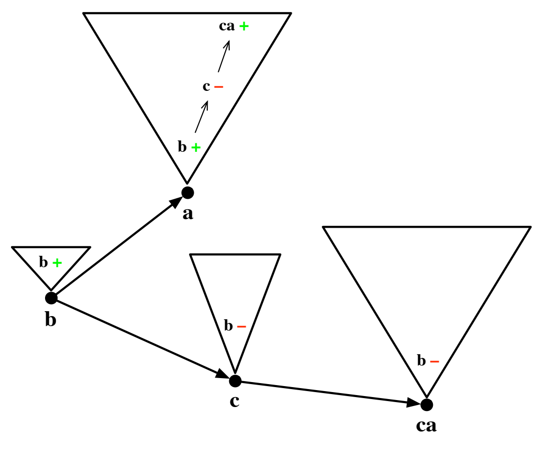

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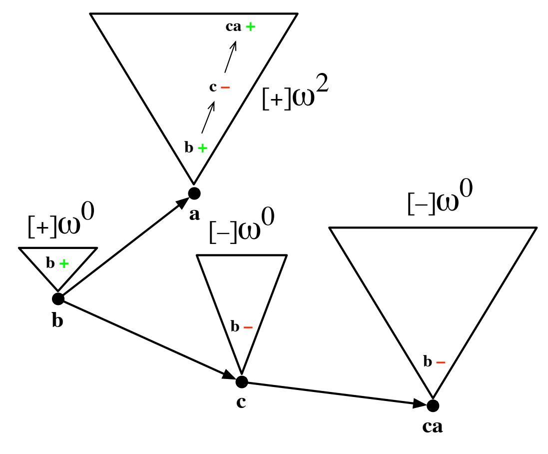
$b^\omega = aa^\omega = a(ca)^\omega = ba^\omega = a^\omega \qquad c^\omega = 0 \qquad b(ca)^\omega = ca^\omega = c(ca)^\omega = (ca)^\omega$

- $Y = \{a^\omega\}$.



Example (continued)

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$
  $a^2 = ab = ac = ba = a \quad b^2 = b \quad bc = cb = c^2 = c$
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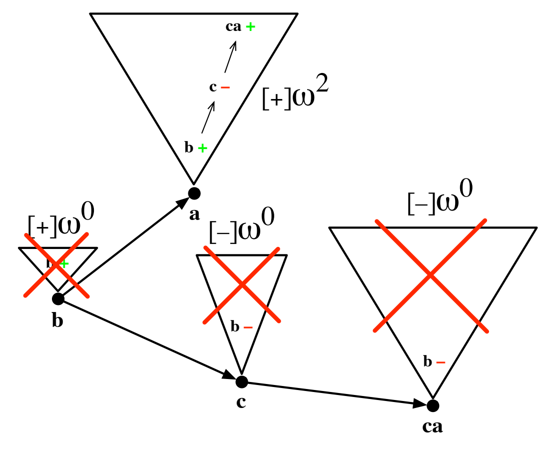

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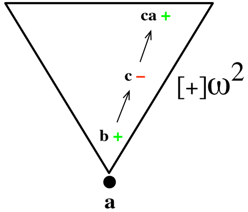

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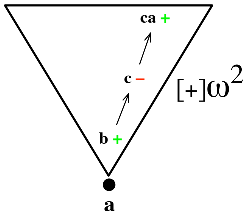


Therefore $d_{SG}((T,Y)) = d_W(L) = \omega^2$,

and L is non-self-dual.

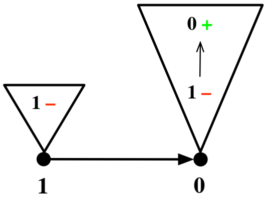
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

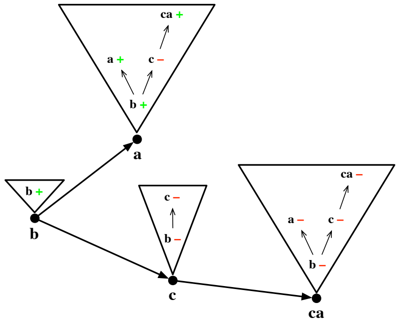
(S,X) I

(T,Y) II

Player II

Player I





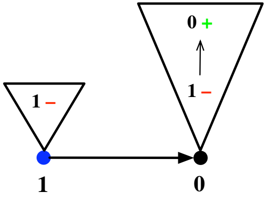
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

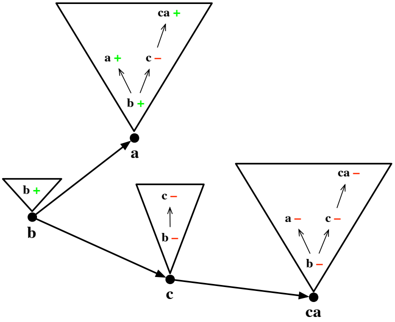
(S,X) **I** **1**

(T,Y) **II**

Player II

Player I



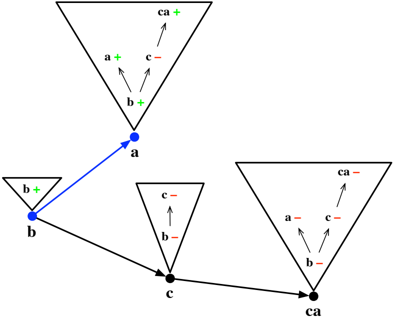


Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

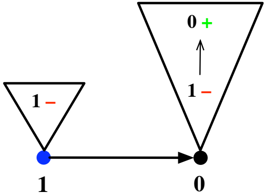
(S,X) **I** **1**

(T,Y) **II** ***a***

Player II



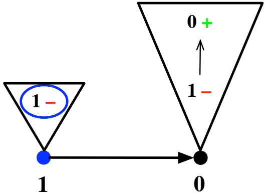
Player I



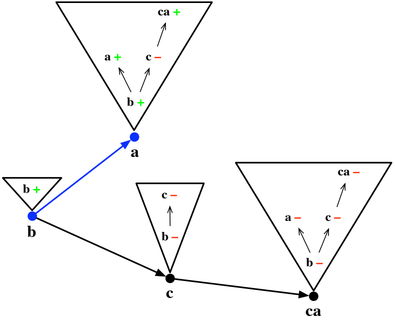
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	

Player I



Player II

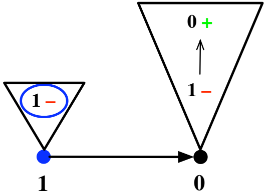


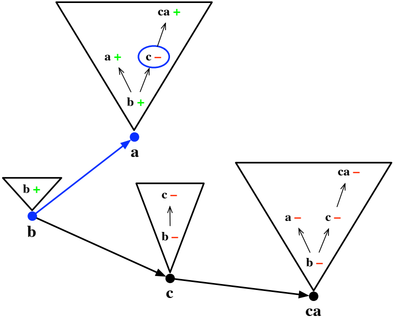
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

Player I

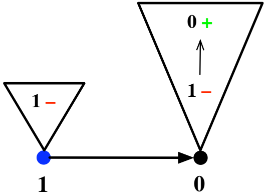




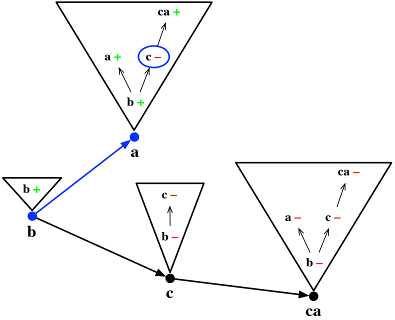
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	a	c

Player I



Player II



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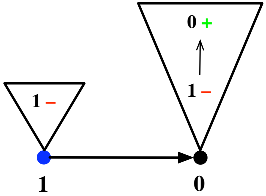
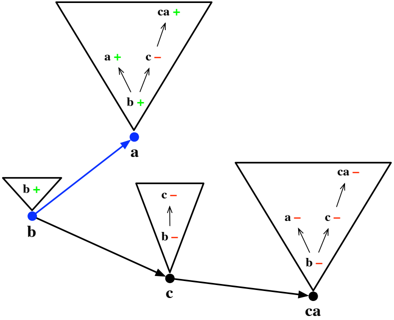
(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player I



Player II



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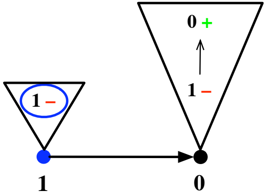
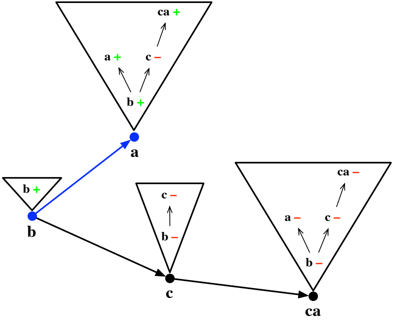
(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	

Player II

Player I





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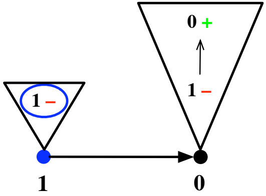
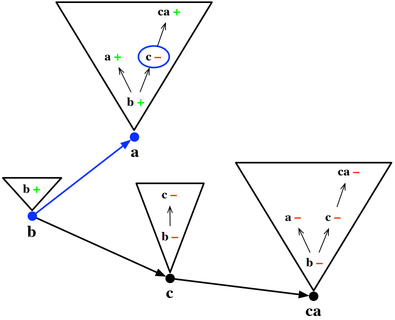
(T,Y)	**II**	***a***	***c***	***c***

Player II

Player I





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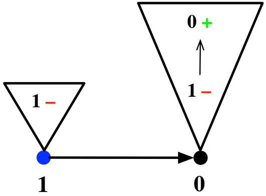
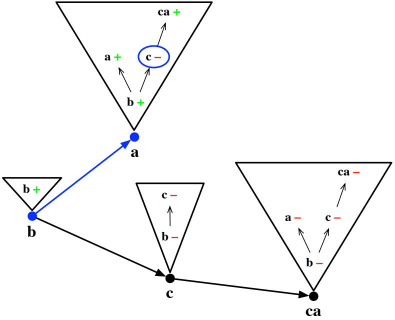
Player I



Player II



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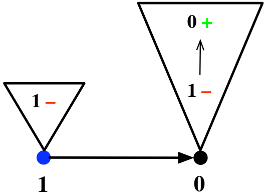
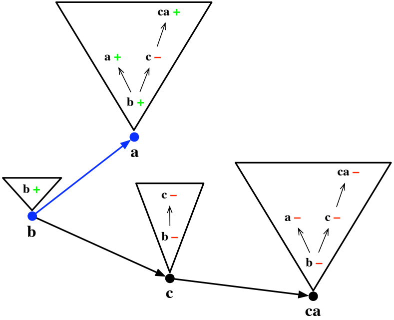
(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player II

Player I





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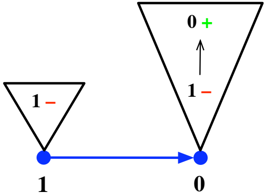
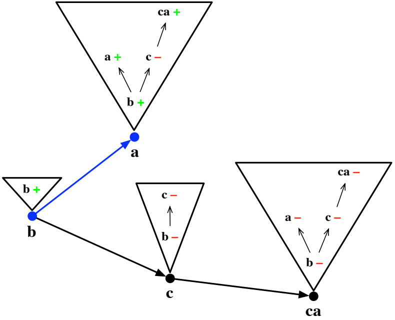
(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	

Player II

Player I





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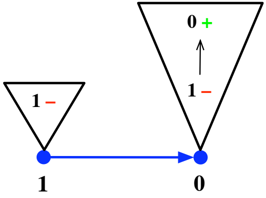
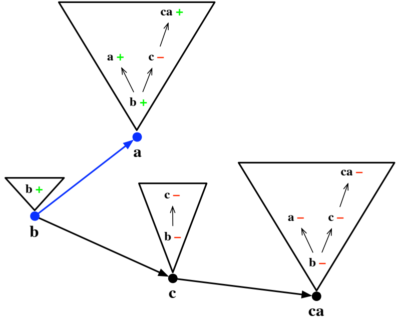
(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**

Player II

Player I





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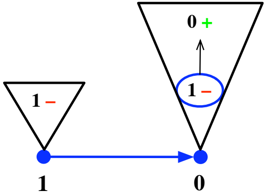
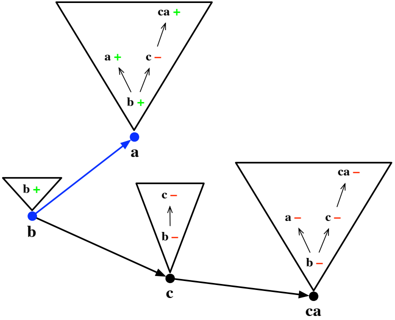
(T,Y)	**II**	***a***	***c***	***c***	**–**	

Player I



Player II



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(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

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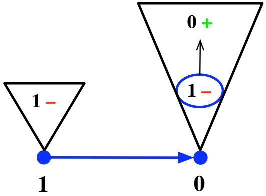
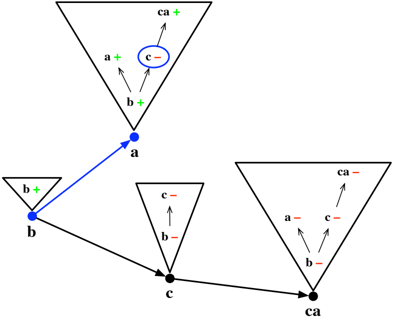


Player II



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(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

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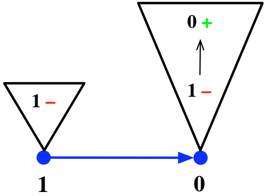

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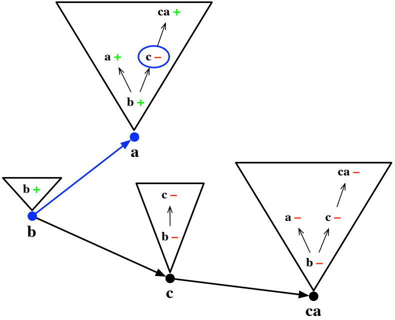

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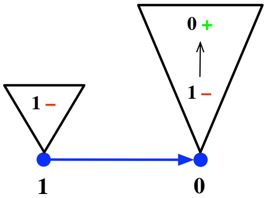
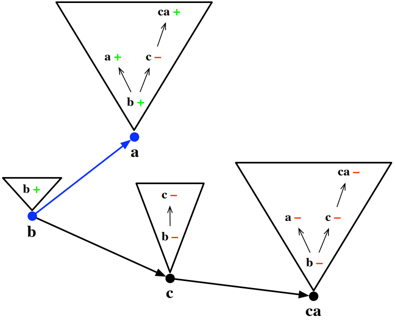
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

Player II

Player I





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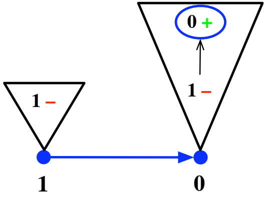
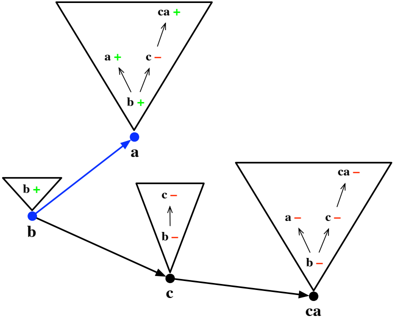
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	

Player II

Player I





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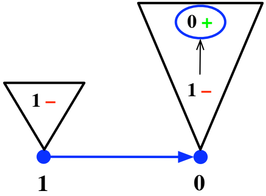
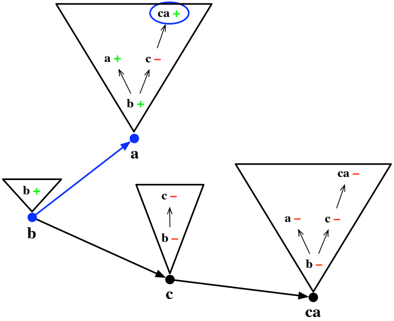
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player I



Player II



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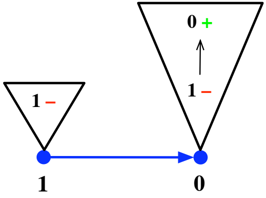
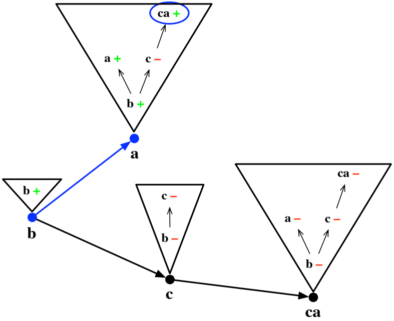
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player II

Player I





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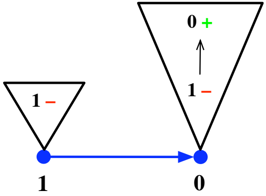
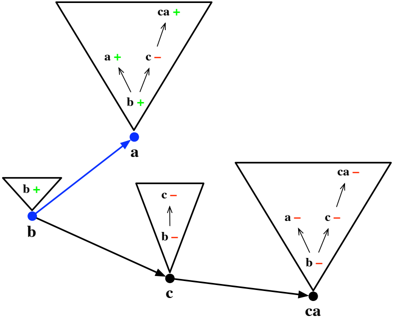
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player II

Player I





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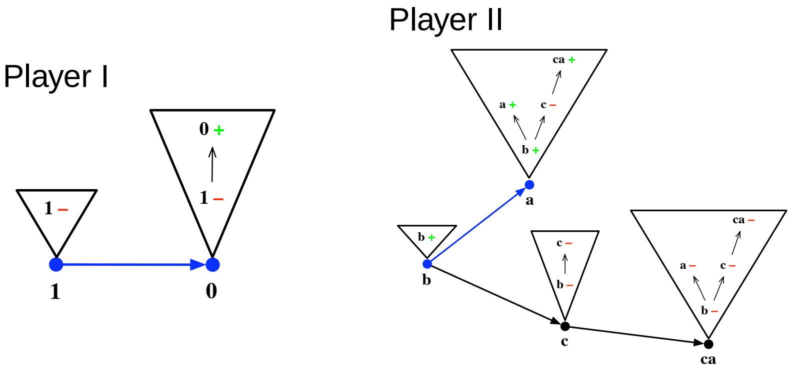
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***	**......**



Summary:

- The $\mathbb{FSG}$-hierarchy is the algebraic counterpart of the Wagner hierarchy.
- Decidability procedure of the $\mathbb{FSG}$-hierarchy.
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