On Interactively Computable Functions

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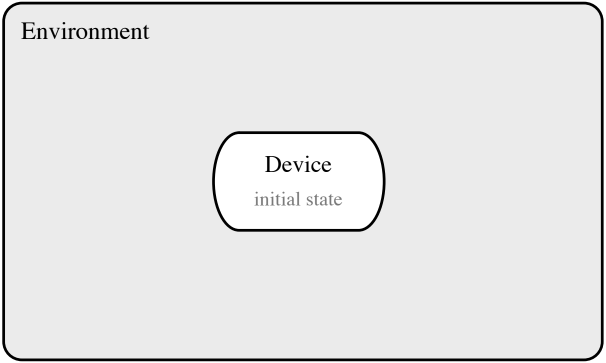
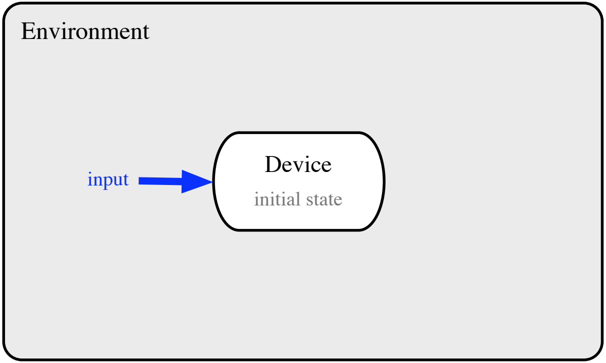
Classical Computation



Closed-box and amnesic...

Classical Computation



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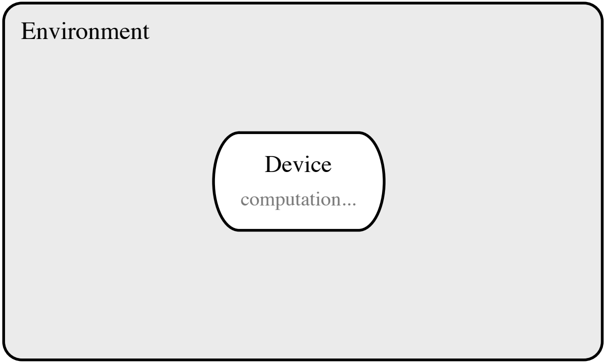

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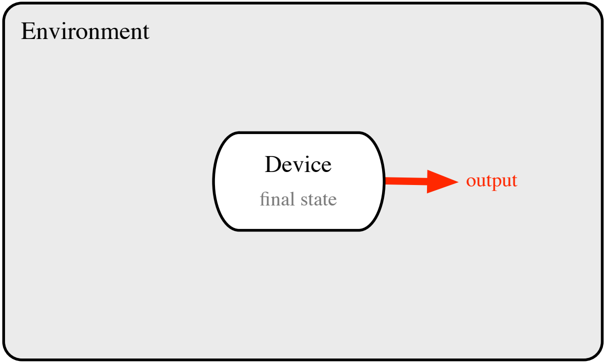

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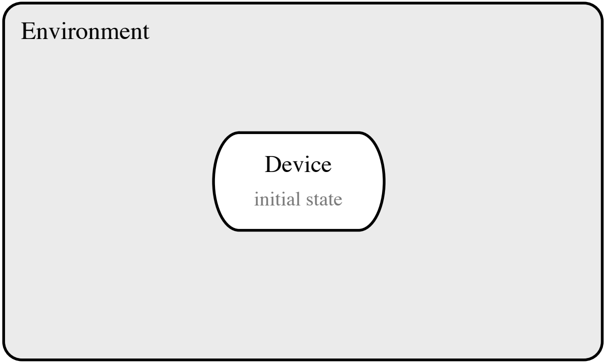

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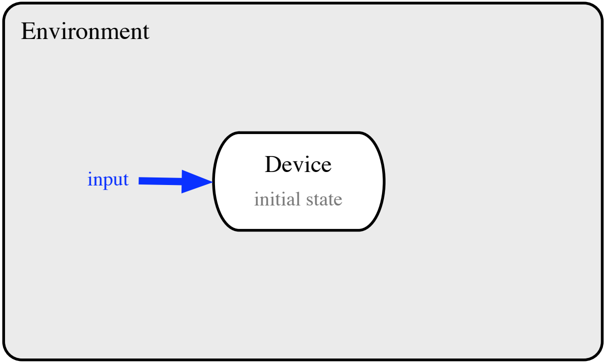

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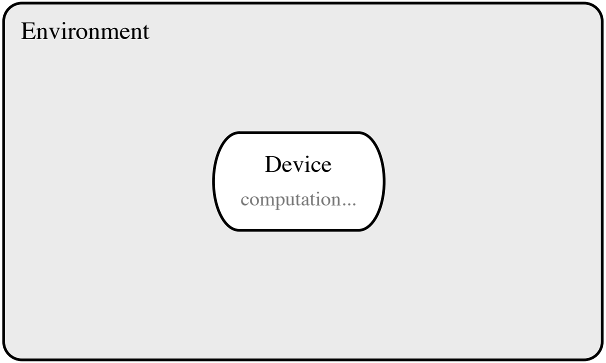

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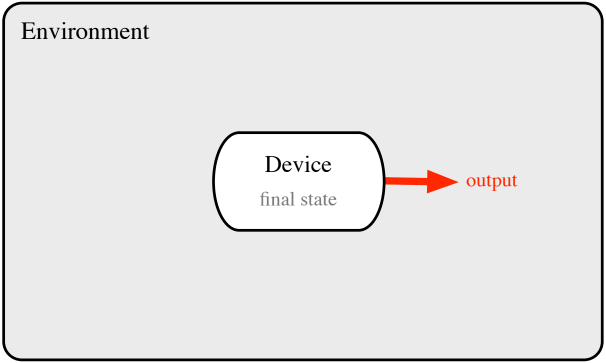

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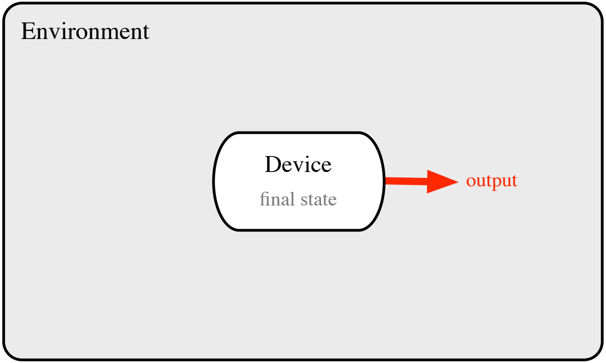

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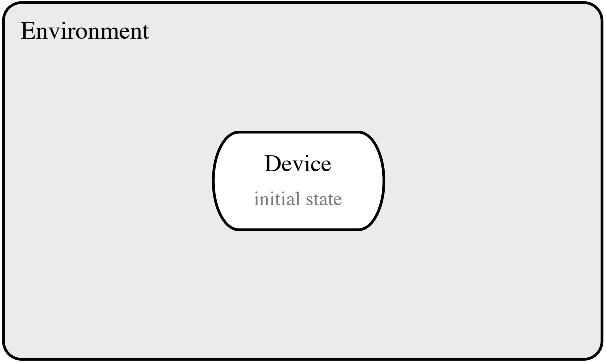

Sequentially interactive and memory active...

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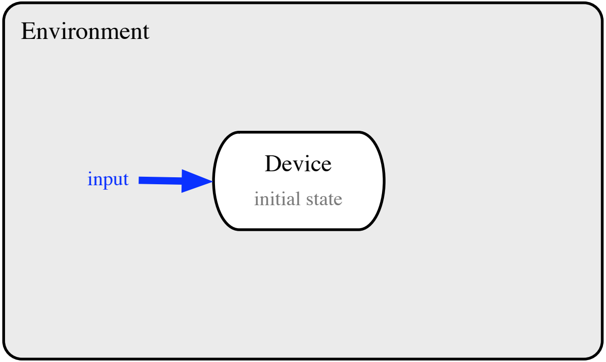

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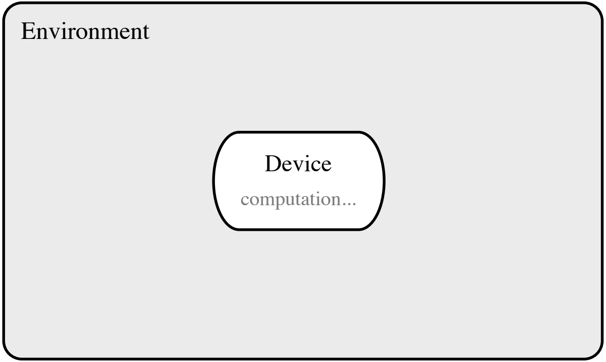

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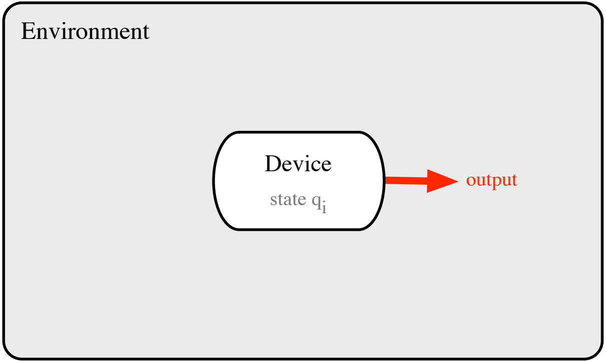

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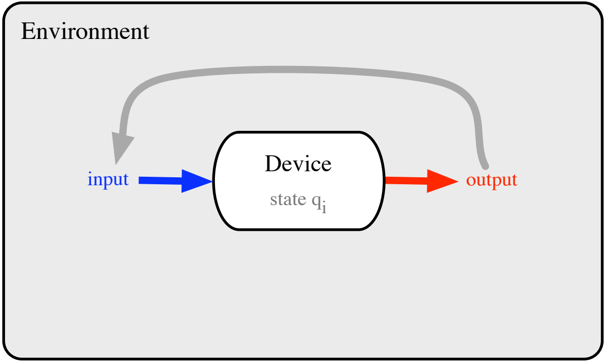

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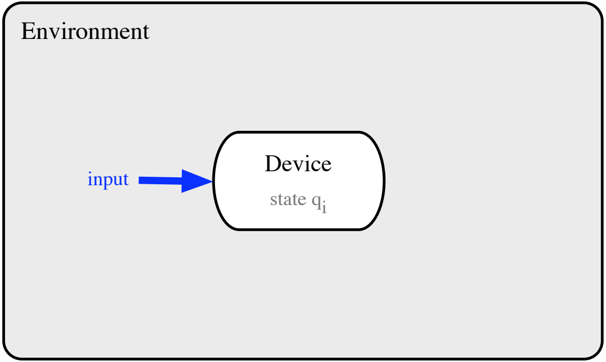

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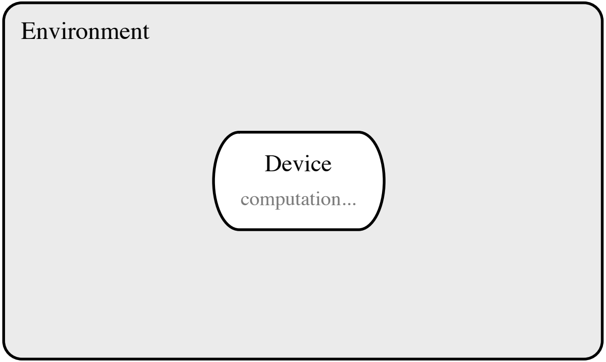

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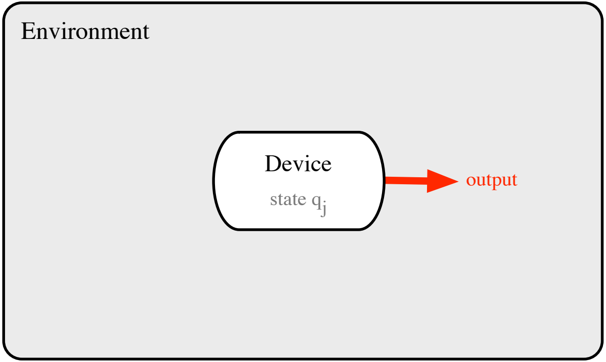

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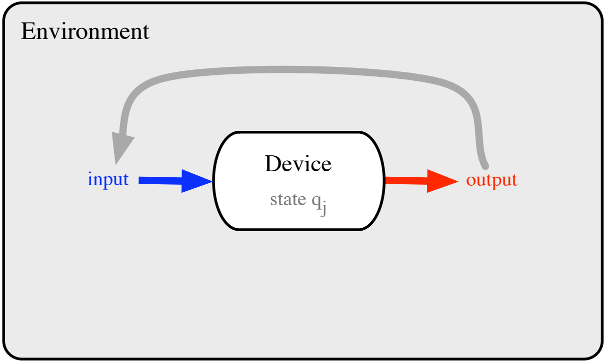

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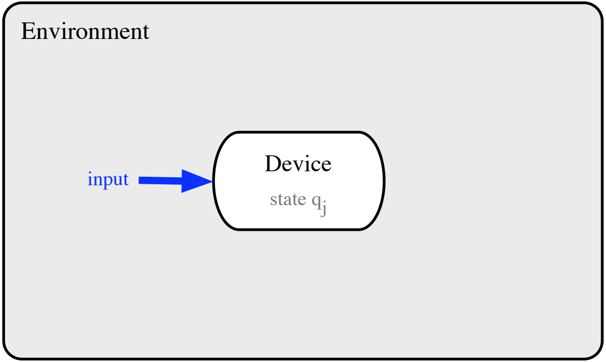

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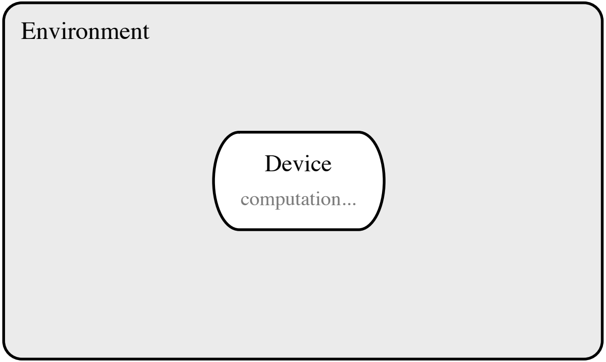

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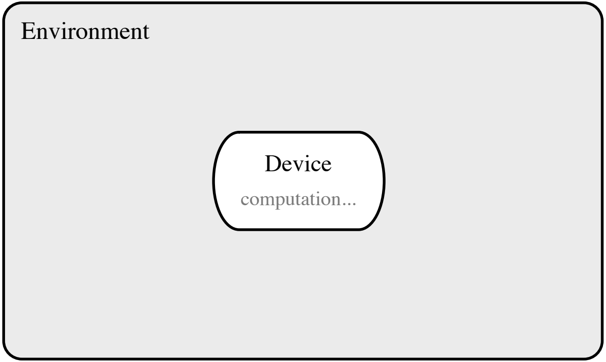

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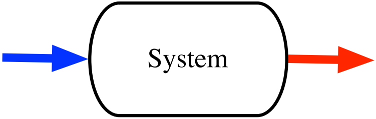

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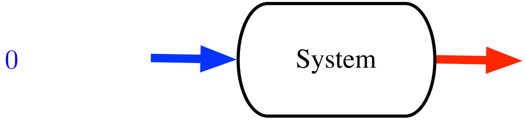

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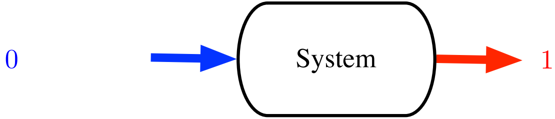

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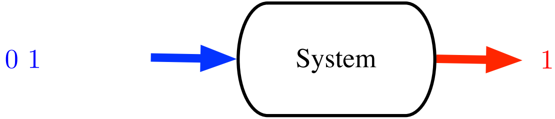

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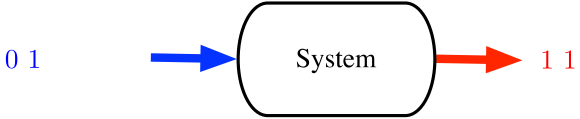

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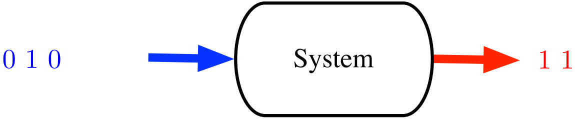

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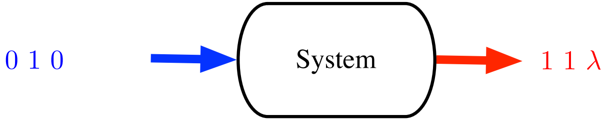

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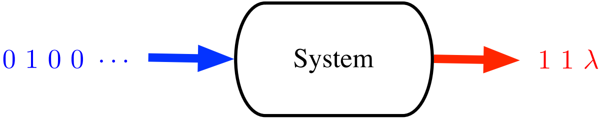

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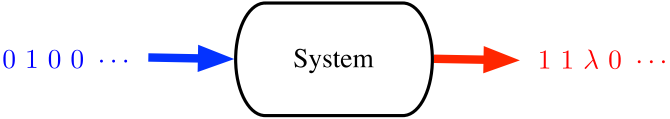

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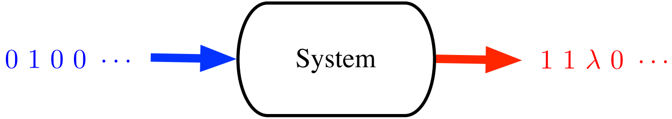

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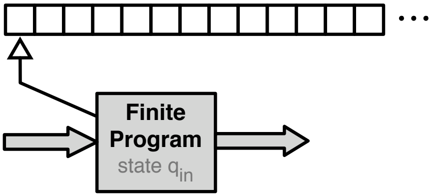
Interactive Turing machine (ITM)



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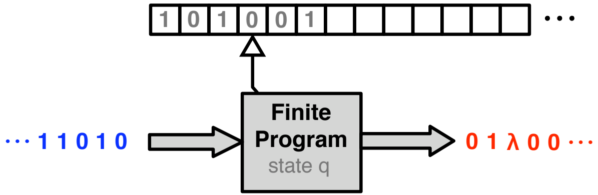
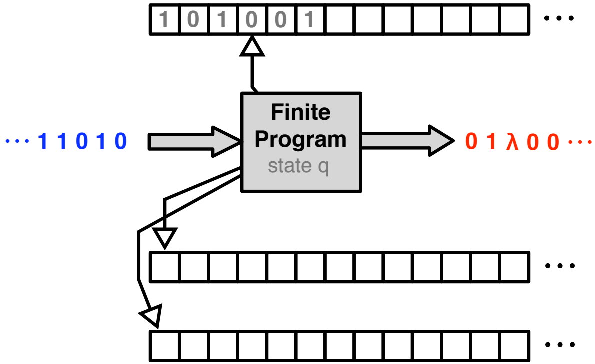


Interactive Turing machine with advice (ITM/A)

An ITM provided with additional advice input and output tapes, as well as an advice function $\alpha : \mathbb{N} \longrightarrow \{0,1\}^*$



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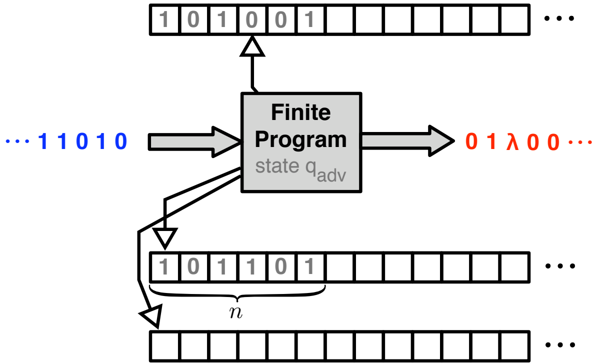

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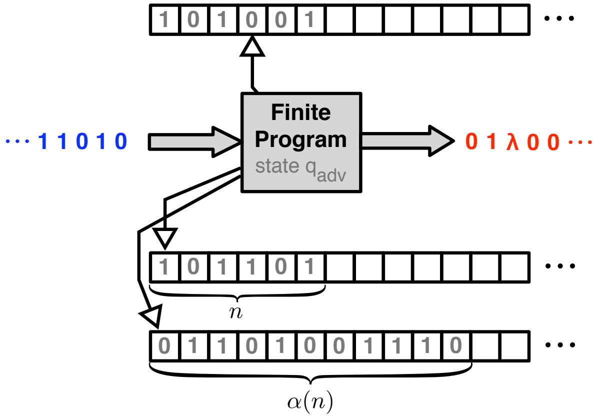

Continuous and Recursive Continuous ω-Translations

Any monotone function $f : \{0,1\}^* \to \{0,1\}^*$ induces *in the limit* an ω-translation $f_\omega : \{0,1\}^\omega \to \{0,1\}^{\leq\omega}$ defined by

$$f_\omega(x) = \lim_{i \geq 0} f(x[0{:}i])$$

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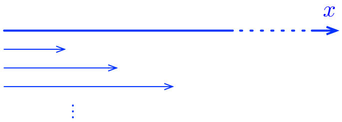

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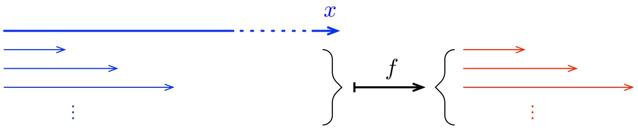

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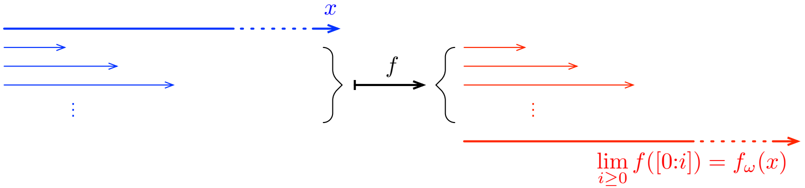

Continuous and Recursive Continuous ω-Translations

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- An ω-translation $\psi : \{0,1\}^\omega \to \{0,1\}^{\leq\omega}$ is called *recursive continuous* if it is induced in the limit by some monotone recursive function, i.e. if there exists some monotone and recursive function $f : \{0,1\}^* \to \{0,1\}^*$ such that $\psi = f_\omega$.

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ITMs and ITM/As Computable Functions

One has the following characterisation of the ω-translations computable by ITM and ITM/A.

Theorem 1

1. *An ω-translation is ITM-computable if an only if it is recursive continuous.*
2. *An ω-translation is ITM/A-computable if an only if it is continuous.*

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Proof: "$(\rightarrow)$"

- Let $\varphi_{\mathcal{S}}$ be some ω-translation computable by some interactive system $\mathcal{S}$.
- Consider $f : \{0,1\}^* \rightarrow \{0,1\}^*$ defined by $f(u) =$ finite word produced by $\mathcal{S}$ after exactly $|u|$ computational steps.
- Then f is monotone and one can show that $f_\omega = \varphi_{\mathcal{S}}$.
- Therefore $\varphi_{\mathcal{S}}$ is continuous.

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- Let ψ be some continuous ω-translation.
- Then there exists some monotone function $f : \{0,1\}^* \to \{0,1\}^*$ such that $f_\omega = \psi$.
- One can show that there exists an ITM/A $\mathcal{M}$ such that $\varphi_{\mathcal{M}} = f_\omega = \psi$.
- Therefore ψ is ITM/A-computable, meaning that ψ is interactively computable. □

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- Therefore ψ is ITM/A-computable, meaning that ψ is interactively computable. □

Interactively Computable Functions

Theorem 2

An ω-translation is interactively computable if and only if it is continuous.

Proof: "$(\leftarrow)$"

- Let ψ be some continuous ω-translation.
- Then there exists some monotone function $f : \{0,1\}^* \to \{0,1\}^*$ such that $f_\omega = \psi$.
- One can show that there exists an ITM/A $\mathcal{M}$ such that $\varphi_{\mathcal{M}} = f_\omega = \psi$.
- Therefore ψ is ITM/A-computable, meaning that ψ is interactively computable. $\square$

Universality of ITM/As

Putting together theorems 2 and 1 point (2), one obtains the universality of ITM/As for interactive computation.

Theorem 3

Let ψ be some interactively computable ω-translation. The following conditions are equivalent:

- *ψ is interactively computable;*
- *ψ is ITM/A-computable;*
- *ψ is continuous.*

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Conclusion

- We provided a machine-based and a mathematical characterisation of the interactively computable functions.
- We showed the model of an ITM/A is universal for interactive computation.
- One can show that evolving recurrent neural networks are also universal for interactive computation (ICANN 2014).
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