Constructing the Algebraic Counterpart of the Wagner Hierarchy by Way of Games

joint work with Jacques Duparc

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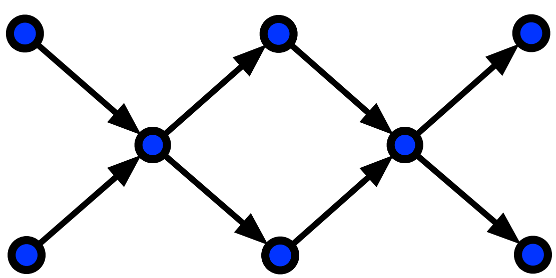
The Wagner Hierarchy
(hierarchy of ω-rational sets)

Let $\mathcal{A}$ and $\mathcal{B}$ be two Muller automata,

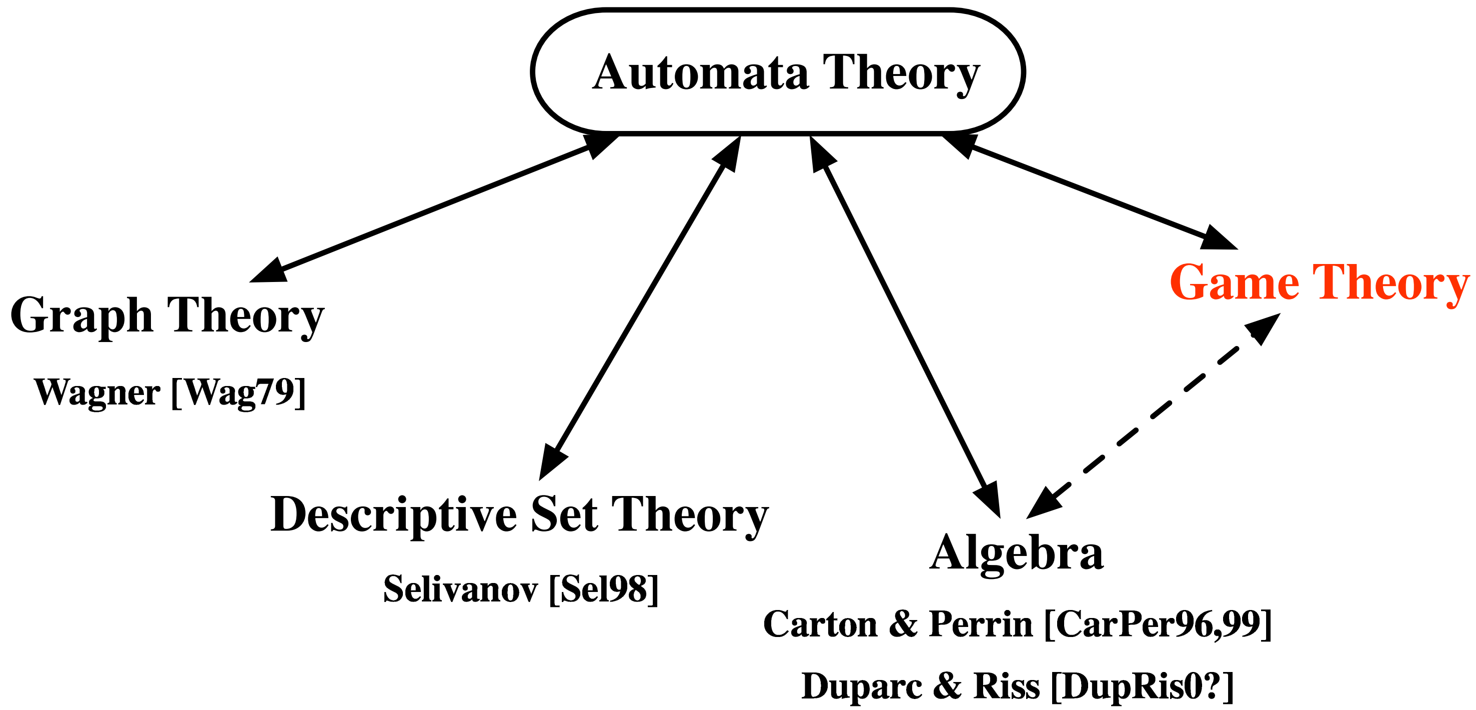
$$\boldsymbol{L(\mathcal{A}) \leq_W L(\mathcal{B})} \;\Leftrightarrow_{def}\; L(\mathcal{A}) \textbf{ is "simpler" than } L(\mathcal{B})$$

$$\textbf{i.e. } \exists\, f \textbf{ cont. s.t. } L(\mathcal{A}) = f^{-1}\big(L(\mathcal{B})\big)$$



• • • height ω^ω

Studies of the Wagner hierarchy



We give a natural game theoretical description...

AUTOMATA THEORY		**ALGEBRA**
automaton **(rational language)**	$\longleftrightarrow$	**finite semigroup**
Büchi automaton **(ω-rational language)**	$\longleftrightarrow$	**Wilke algebra** **finite $\boldsymbol{\omega}$-semigroup**

ω-*semigroup* $S = (S_+, S_\omega)$ (J.-É. Pin)

- $(S_+, \cdot)$ is a semigroup, S_ω is a set
- $\pi : {S_+}^\omega \longrightarrow S_\omega$ an infinite product

A reduction relation $\leq_{SG}$ **on** ω**-semigroups**

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$

$\boldsymbol{X} \leq_{SG} \boldsymbol{Y}$ $\Leftrightarrow_{def}$ X **is "less complicated" than** Y

i.e. $\exists$ **"simple"** f **s.t.** $(u \in X \Leftrightarrow f(u) \in Y)$

$\Leftrightarrow_{def}$ **II has a w.s. in the game** $\mathbb{SG}(X, Y)$

An infinite two-player game $\mathbb{SG}(\boldsymbol{X}, \boldsymbol{Y})$ *on* ω*-semigroups*

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$**.**

(X) **I**	s_0	s_1	$\ldots$		$\xrightarrow{\textbf{after } \omega \textbf{ moves}}$	$\langle s_0, s_1, s_2, \ldots \rangle$
(Y) **II**		t_0	t_1	$\ldots$	$\xrightarrow{\textbf{after } \omega \textbf{ moves}}$	$\langle t_0, t_1, t_2, \ldots \rangle$

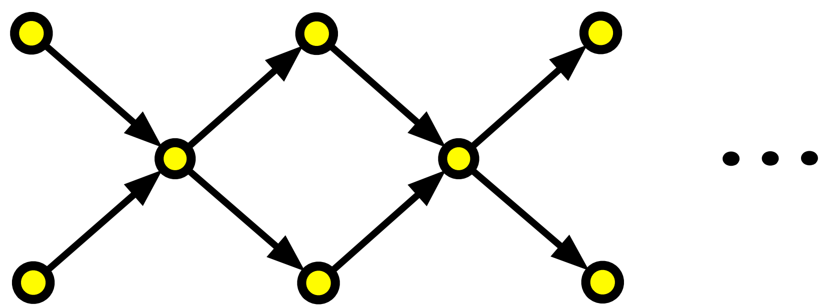
II wins

$$\Leftrightarrow_{def}$$

$$\boldsymbol{\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y}$$

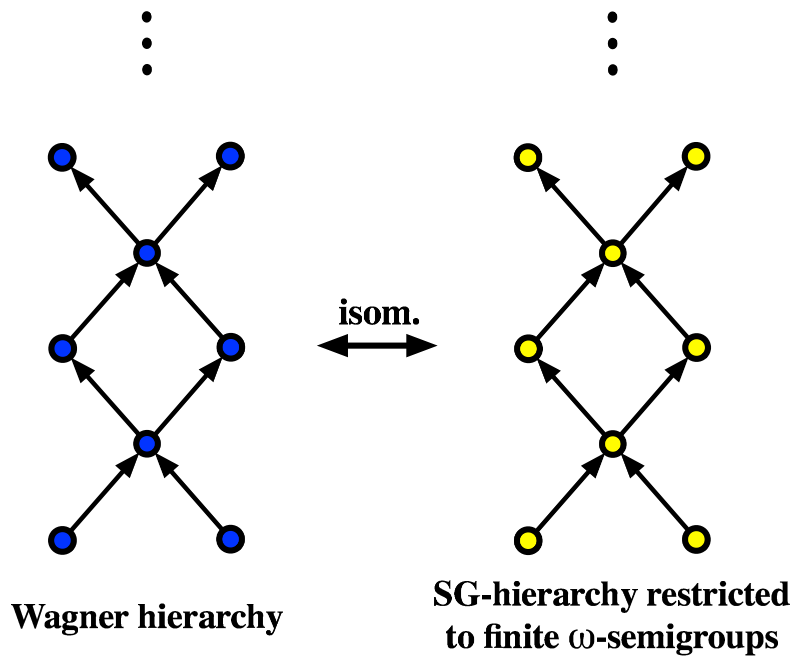
The $\mathbb{SG}$*-hierarchy*

The $\leq_{SG}$ induces a hierarchy on ω-subsets



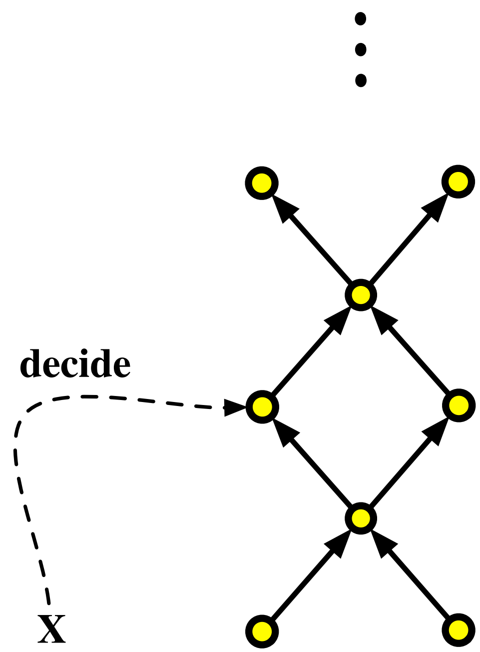
Proposition

The finite $\mathbb{SG}$-hierarchy is classwise isomorphic to the Wagner hierarchy.



Proposition

The finite $\mathbb{SG}$-hierarchy is decidable: given $X \subseteq S_\omega$, we can compute its degree ξ_X in the $\mathbb{SG}$-hierarchy



Linked pairs

Let S_+ be a semigroup,

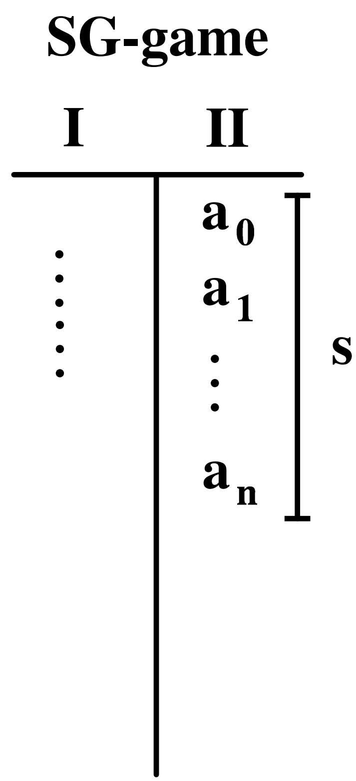
$$(s,e) \in S_+ \times S_+ \textbf{ is a } \textbf{\textit{linked pair}} \textbf{ if}$$

1. $se = s$

2. e **is idempotent (i.e.** $e^2 = e$**)**

Linked pair $\equiv$ stable postion in $\mathbb{SG}$-game

Let (s, e) **be a linked pair (so** $se = s$ **and** $e^2 = e$**),**



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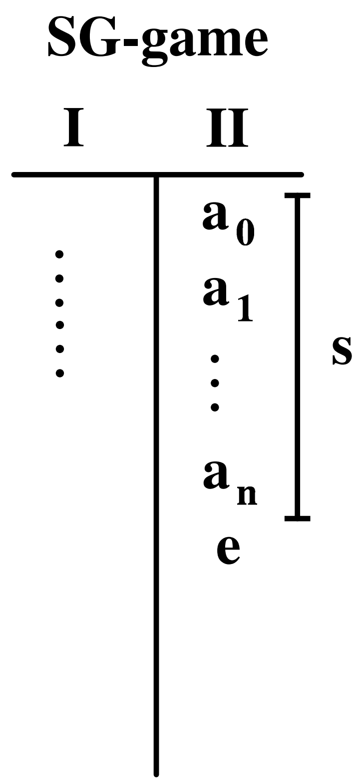

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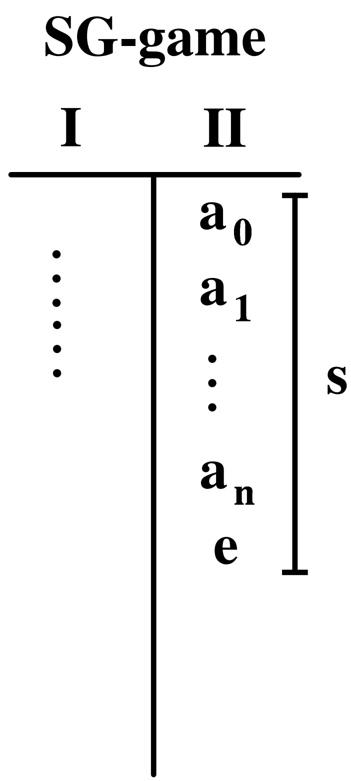

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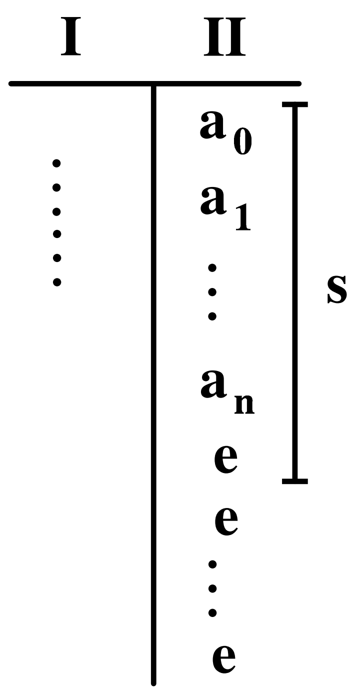

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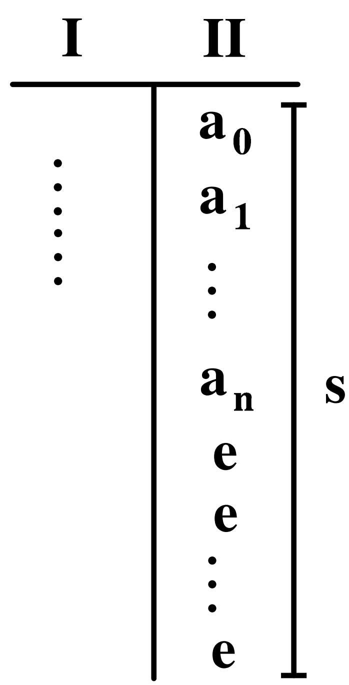


$\boldsymbol{Accessibility\ relation} \ {\le_{acc}}$ between prefixes of linked pairs

Let (s,e) **and** (s',e') **be two linked pairs,**

$$\boldsymbol{s \le_{acc} s'} \quad \Leftrightarrow_{def} \quad \exists\, t \in S_+ \ \textbf{s.t.}\ st = s'$$

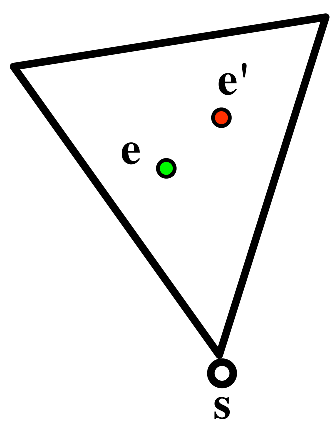
$\Leftrightarrow_{def}$ **a player can go from position** s **to position** s' **in a** $\mathbb{SG}$**-game**

$$\boldsymbol{s \equiv_{acc} s'} \quad \Leftrightarrow_{def} \quad s \le_{acc} s' \le_{acc} s$$

Coloring idempotents of linked pairs with respect to an ω-subset X

Let $S = (S_+, S_\omega)$ **be an** ω**-semigroup,** (s, e) **be a linked pair, and** $X \subseteq S_\omega$**,**

$$e \textbf{ is } \begin{cases} \textbf{green if } \pi(s, e, e, \ldots) \in X \\ \textbf{red if } \pi(s, e, e, \ldots) \notin X \end{cases}$$



Colored tree-representation of an ω-subset X

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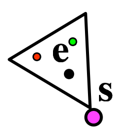

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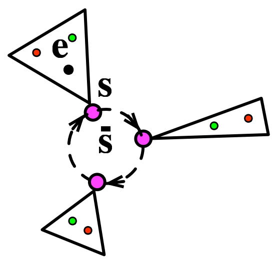

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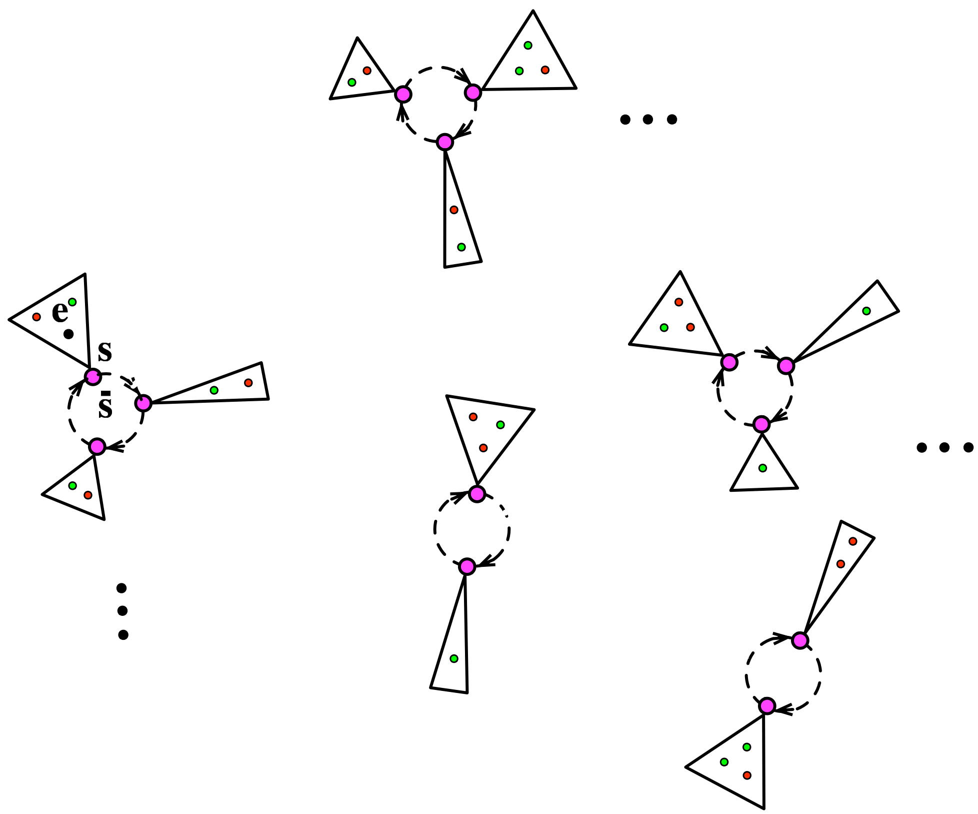

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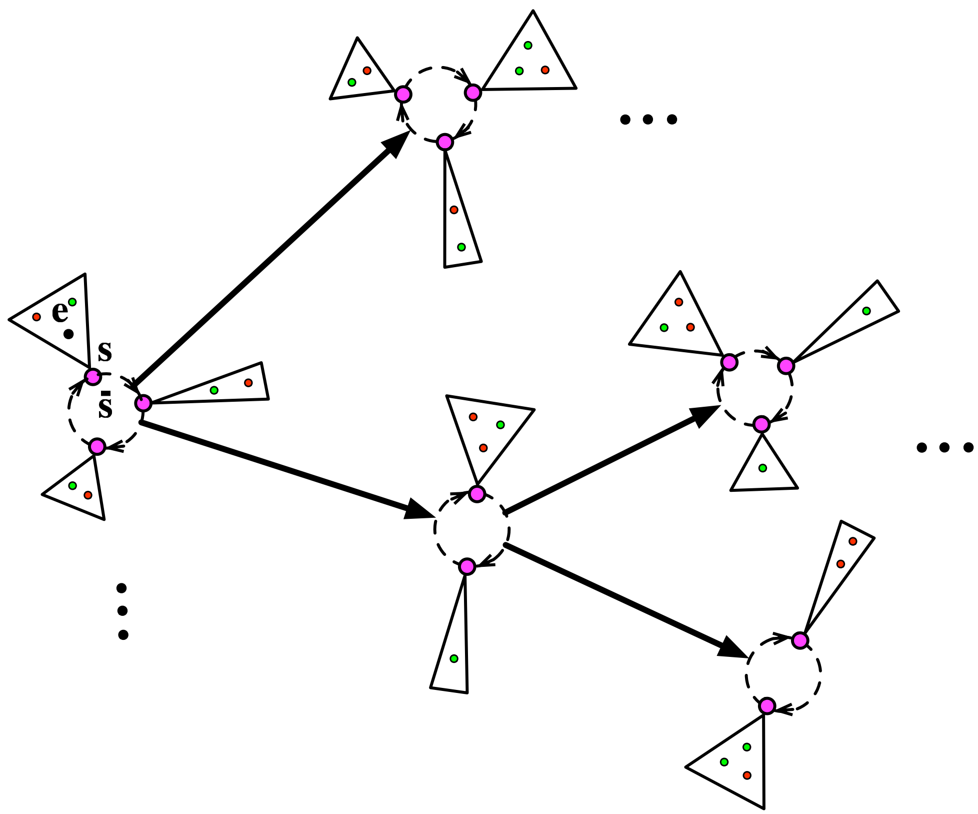

$\Rightarrow$ **algorithm deciding the** $\mathbb{SG}$**-degree of** X **follows.**

Conclusions

- **Develop interactions between Algebra and Game Theory**
- **Try to find the algebraic correspondence of other machines, and characterize the hierarchy of their languages by this method.**