Expressive power of Recurrent Neural Networks over Infinite Input Streams

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Université Paris 2

27 May 2016

Introduction

- We assume that some aspects of information processing in the brain can be approached from the perspective of computability theory.
- The computational capabilities of recurrent neural networks have mainly been studied in the context of classical computation (McCulloch & Pitts, Turing, Kleene, von Neumann, Minsky, Papert,..., Siegelmann & Sontag,...).
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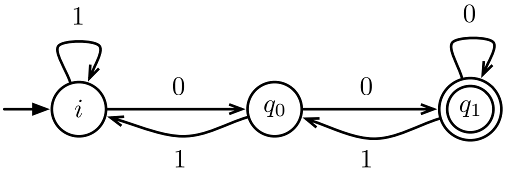
Finite State Automaton



- an input u is *accepted* by $\mathcal{A}$ if $\mathcal{A}(u)$ reaches a final state
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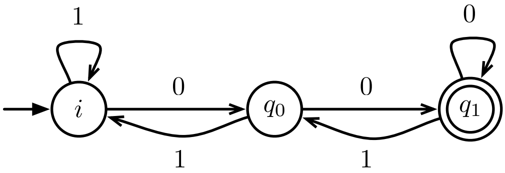

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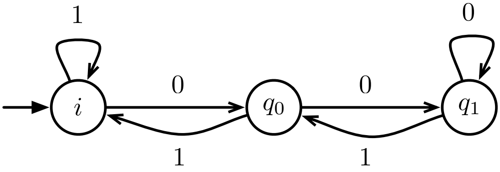

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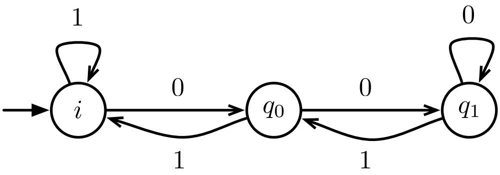

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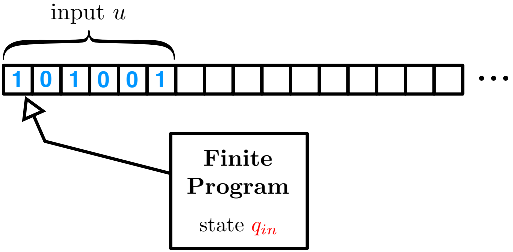
Turing Machine

A *Turing machine* (TM) consists of an infinite tape, a read-write head, and a finite program.



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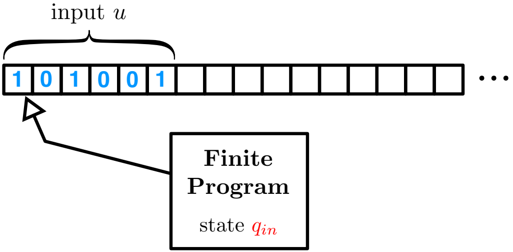

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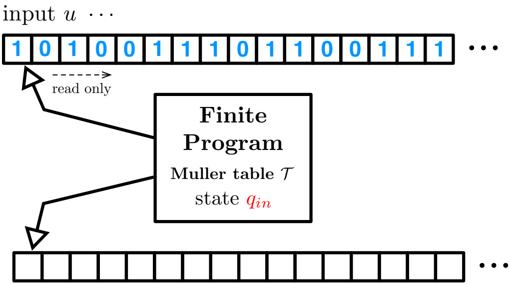
A *Muller Turing machine* consists of a classical TM with Muller acceptance condition.



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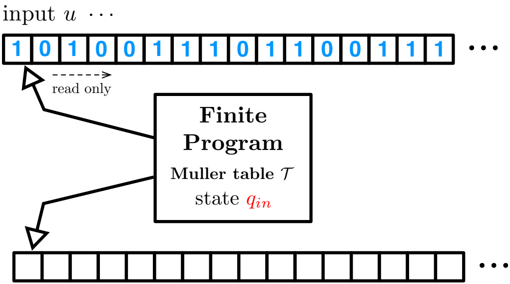

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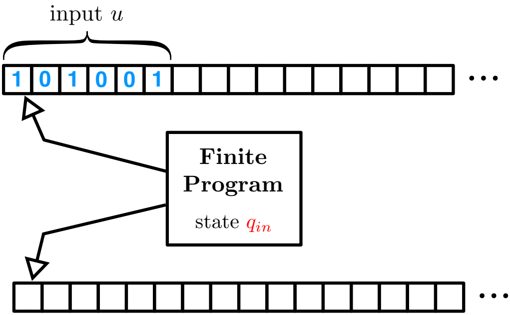
Turing Machine with Advice

A *Turing machine with advice* (TM/A) is a TM provided with an additional advice tape and advice function $\alpha : \mathbb{N} \longrightarrow \{0, 1\}^*$.



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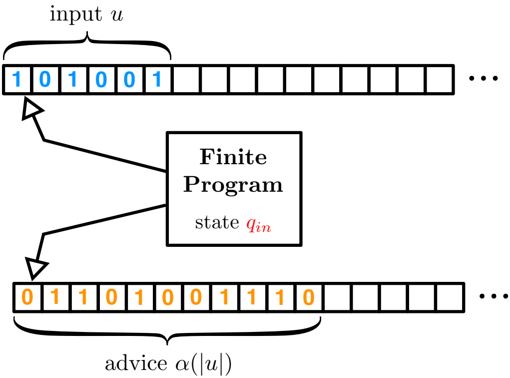

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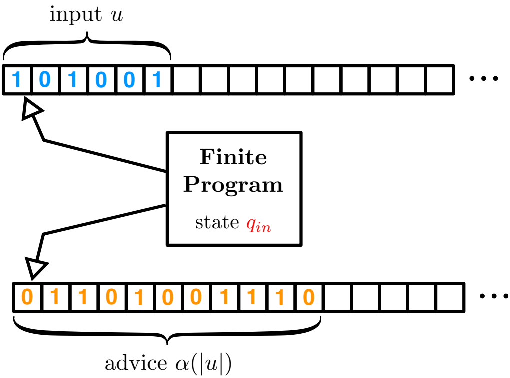

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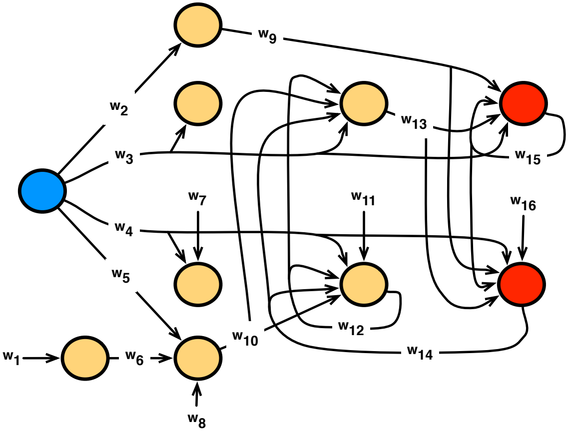
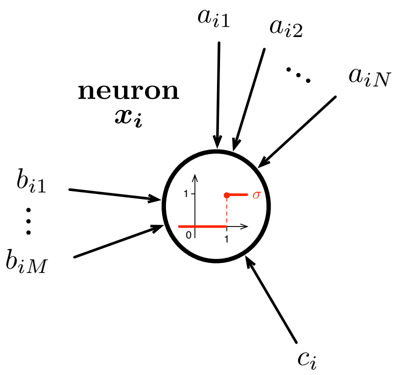
Recurrent Neural Network



Boolean Recurrent Neural Networks



$$x_i(t+1) = \theta\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

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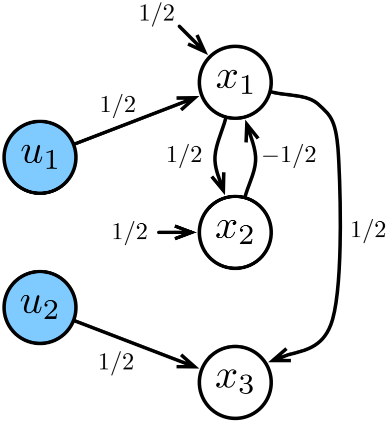

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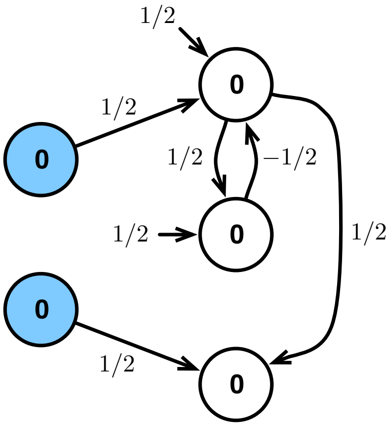

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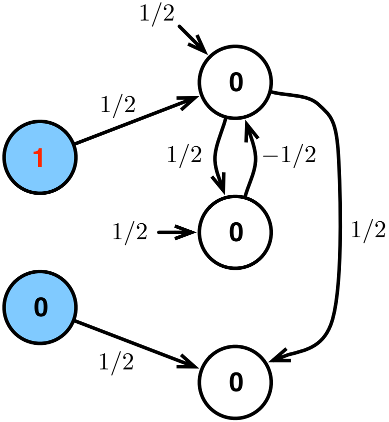

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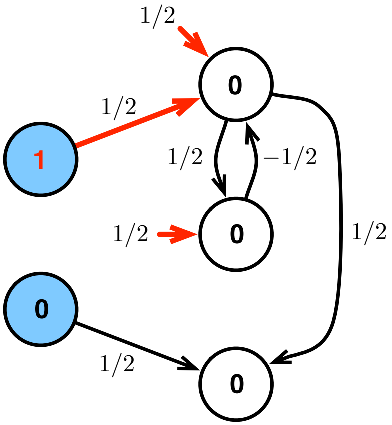

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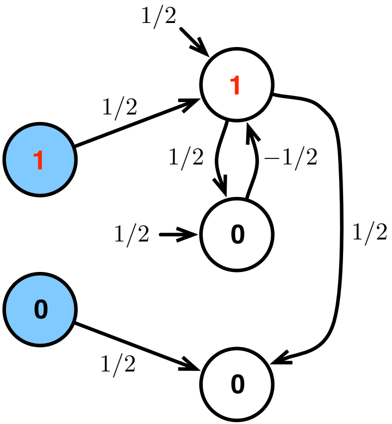

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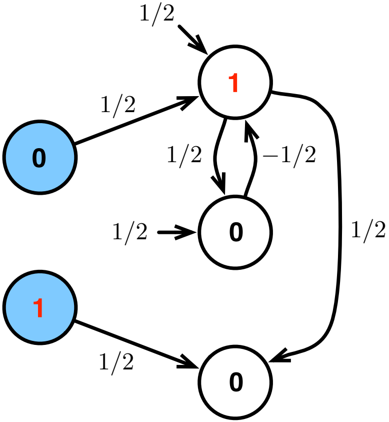

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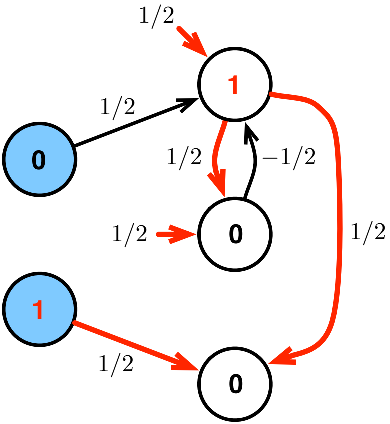

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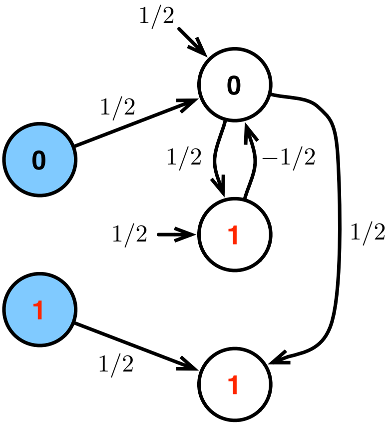
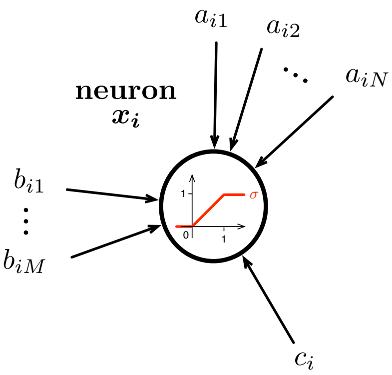


Sigmoidal Recurrent Neural Networks



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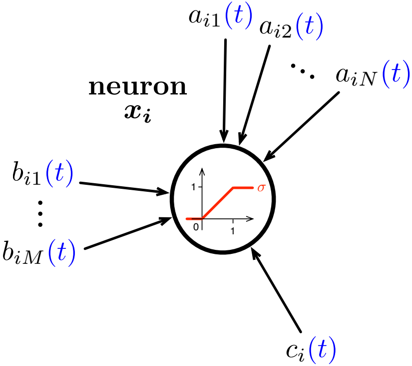

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Recurrent Neural Networks

We consider eight models of RNNs:

1. Boolean rational RNNs: B-RNN[$\mathbb{Q}$]s
2. Boolean real RNNs: B-RNN[$\mathbb{R}$]s
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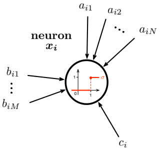

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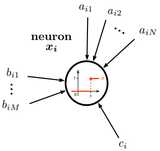

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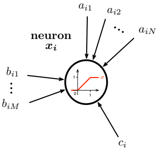

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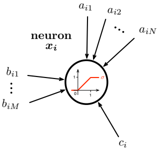

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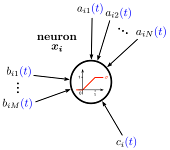

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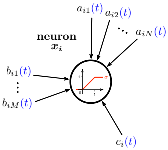

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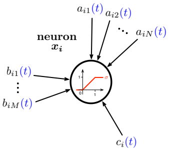

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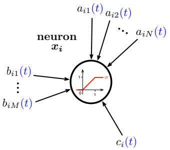

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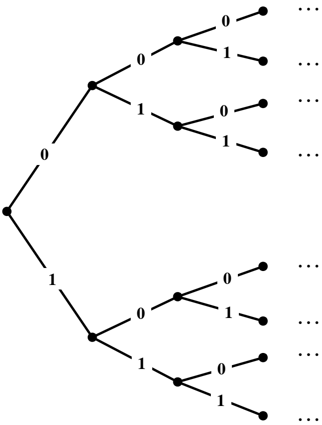
Results (Classical Computation)

	Boolean	Static	Bi-valued Evolving	Evolving
$\mathbb{Q}$	FSA **REG** Kl 56, Mi 67	TM **P** S&S 95	TM/poly(A) **P/poly** C&S 11,14	TM/poly(A) **P/poly** C&S 11,14
$\mathbb{R}$	FSA **REG** Kl 56, Mi 67	TM/poly(A) **P/poly** S&S 94	TM/poly(A) **P/poly** C&S 11,14	TM/poly(A) **P/poly** C&S 11,14

Topology

The Cantor space $\{0,1\}^\omega$

the set of infinite sequences of bits



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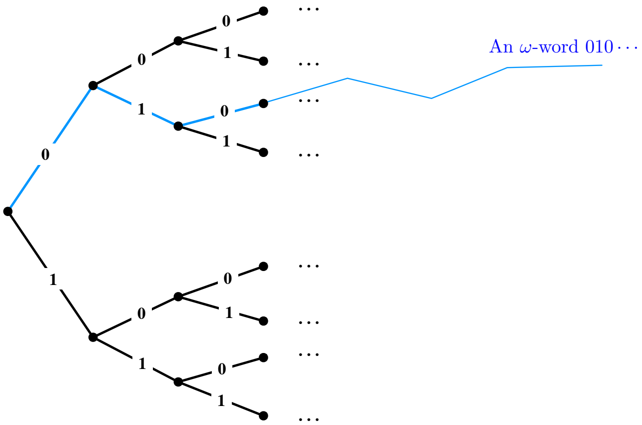

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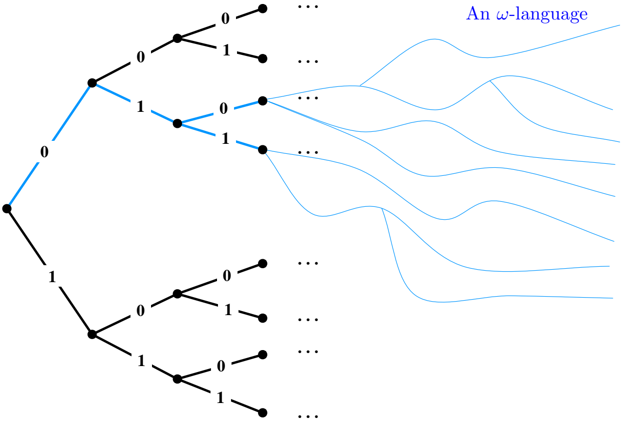

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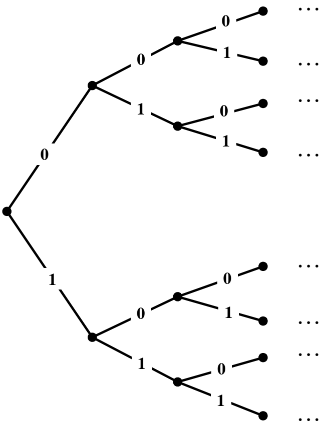

equipped with product topology
of the discrete topology on $\{0,1\}$

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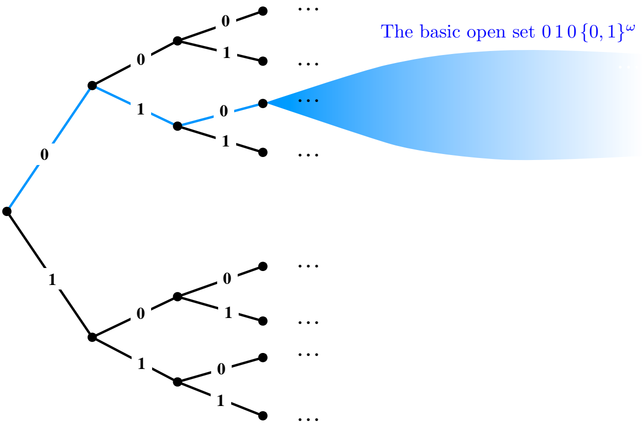

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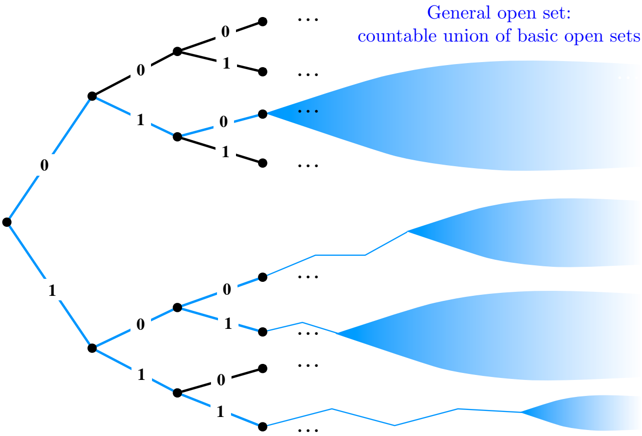
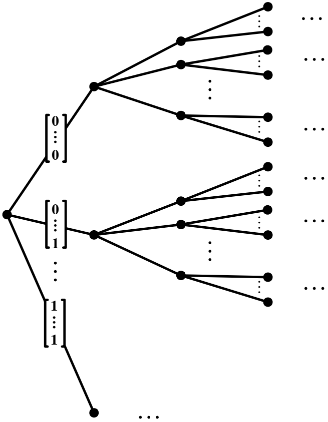
The Cantor space $\{0,1\}^\omega$
the set of infinite sequences of bits



Topology

The space $(\mathbb{B}^N)^\omega$

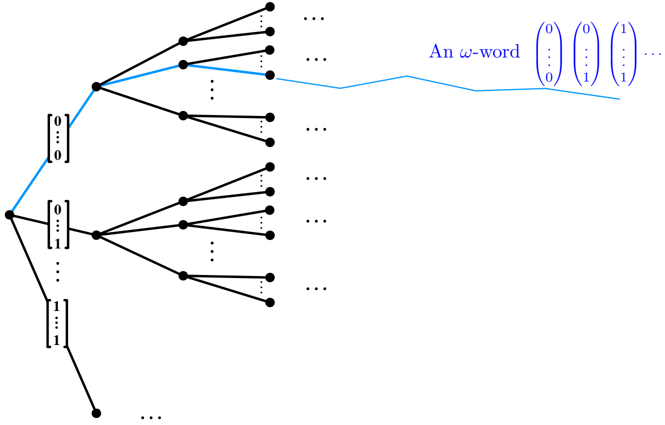
the set of infinite sequences of N-dim. Boolean vectors



Topology

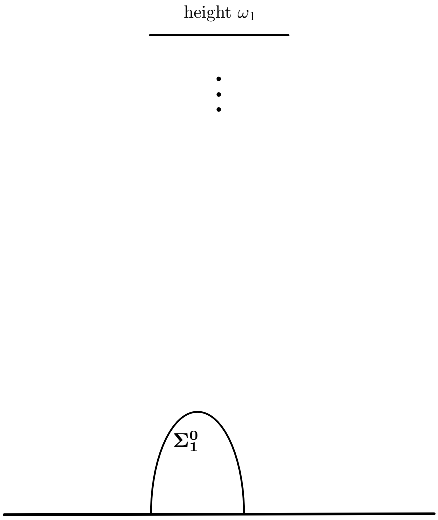
The space $(\mathbb{B}^N)^\omega$

the set of infinite sequences of N-dim. Boolean vectors

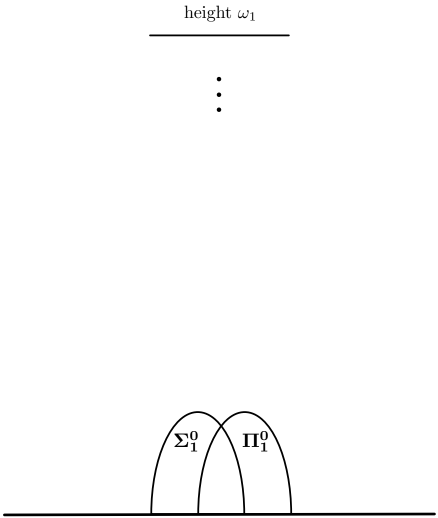


Topology

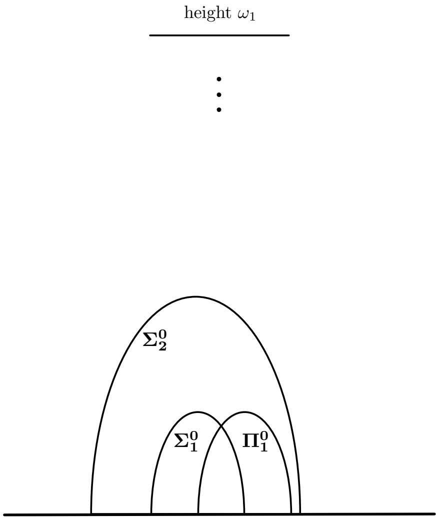
Topology



Topology

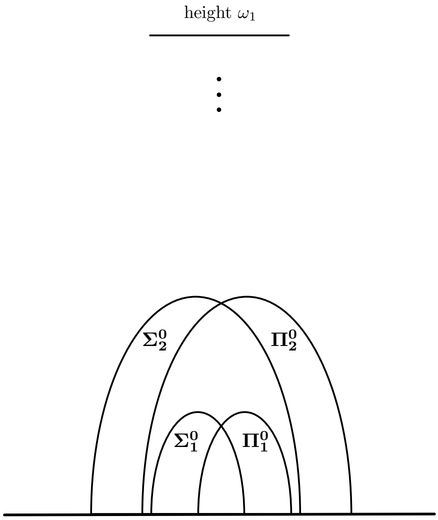


Topology

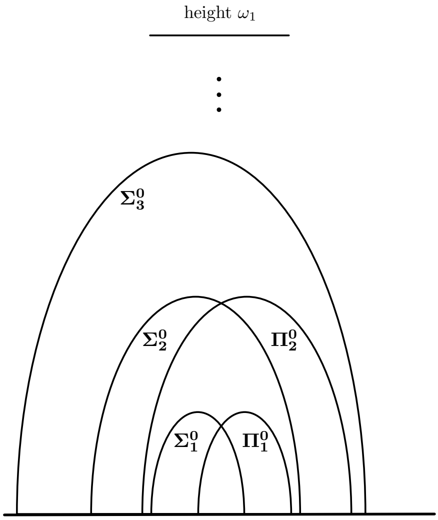


height ω_1
$\vdots$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$

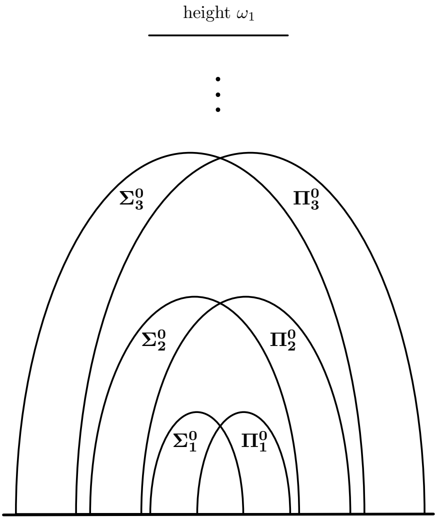
Topology


height ω_1
$\vdots$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

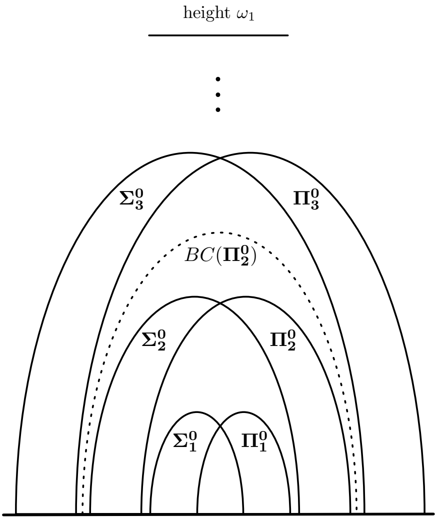
Topology


height ω_1
$\boldsymbol{\Sigma^0_3}$
$\boldsymbol{\Sigma^0_2}$
$\boldsymbol{\Pi^0_2}$
$\boldsymbol{\Sigma^0_1}$
$\boldsymbol{\Pi^0_1}$

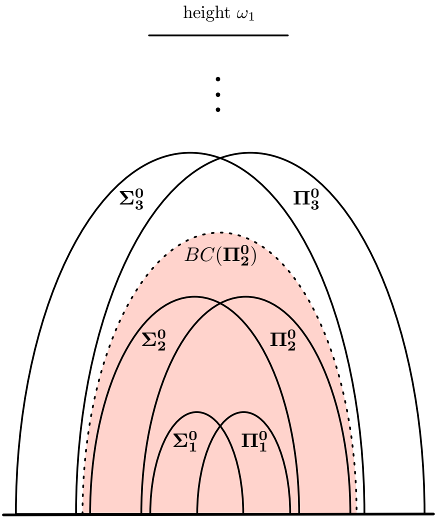
Topology


height ω_1
$\mathbf{\Sigma^0_3}$
$\mathbf{\Pi^0_3}$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

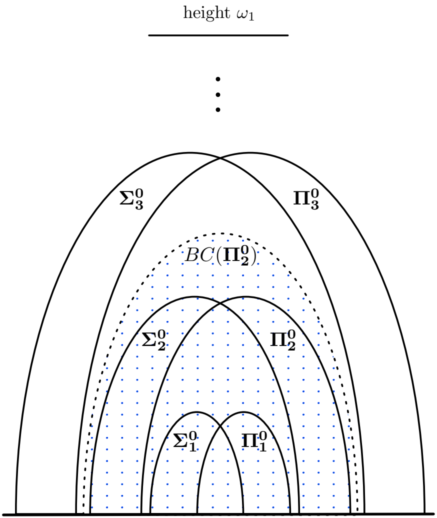
Topology


height ω_1
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$BC(\mathbf{\Pi}^0_2)$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$

Topology


height ω_1
$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Pi}^0_3$
$BC(\boldsymbol{\Pi}^0_2)$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$

Topology

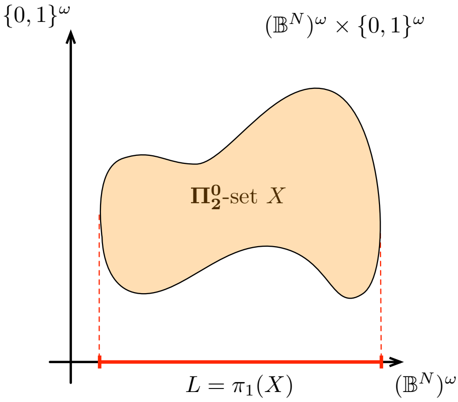

height ω_1
$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Pi}^0_3$
$BC(\boldsymbol{\Pi}^0_2)$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$

Topology

$$
\begin{tikzcd}[column sep=small]
& \vdots & \\
\boldsymbol{\Sigma^0_3} & & \boldsymbol{\Pi^0_3} \\
& \boldsymbol{\Delta^0_3} \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma^0_2} \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi^0_2} \arrow[ul, phantom, sloped, "\supsetneq"] \\
& \boldsymbol{\Delta^0_2} \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] & \\
\boldsymbol{\Sigma^0_1} \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi^0_1} \arrow[ul, phantom, sloped, "\supsetneq"] \\
& \boldsymbol{\Delta^0_1} \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}
$$

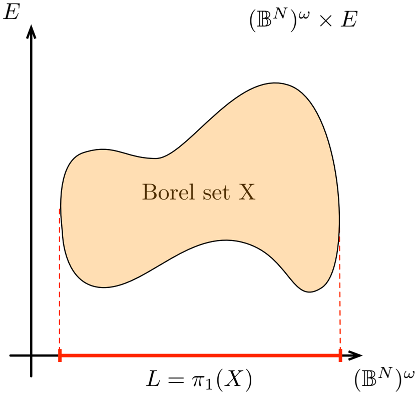
Topology

- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\mathbf{\Sigma}^1_1$) iff it is the first projection of some $\mathbf{\Pi}^0_2$-set $X \subseteq (\mathbb{B}^N)^\omega \times \{0,1\}^\omega$.



Topology

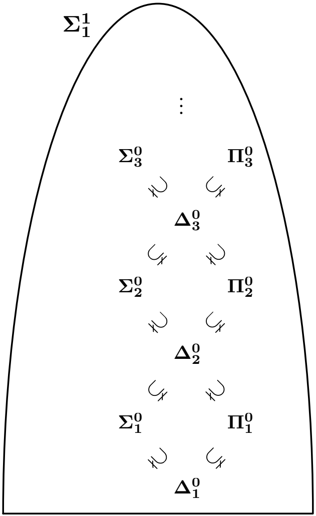
- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\mathbf{\Sigma}_1^1$) iff it is the first projection of some Borel set $X \subseteq (\mathbb{B}^N)^\omega \times E$, where E is a Polish space.



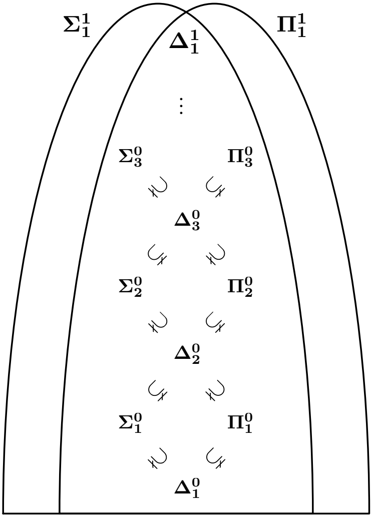
Topology

$$
\begin{tikzcd}
 & \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
 & \boldsymbol{\Delta}^0_3 \arrow[ul, hook, "/" marking] \arrow[ur, hook, "/" marking] & \\
\boldsymbol{\Sigma}^0_2 \arrow[ur, hook, "/" marking] & & \boldsymbol{\Pi}^0_2 \arrow[ul, hook, "/" marking] \\
 & \boldsymbol{\Delta}^0_2 \arrow[ul, hook, "/" marking] \arrow[ur, hook, "/" marking] & \\
\boldsymbol{\Sigma}^0_1 \arrow[ur, hook, "/" marking] & & \boldsymbol{\Pi}^0_1 \arrow[ul, hook, "/" marking] \\
 & \boldsymbol{\Delta}^0_1 \arrow[ul, hook, "/" marking] \arrow[ur, hook, "/" marking] &
\end{tikzcd}
$$

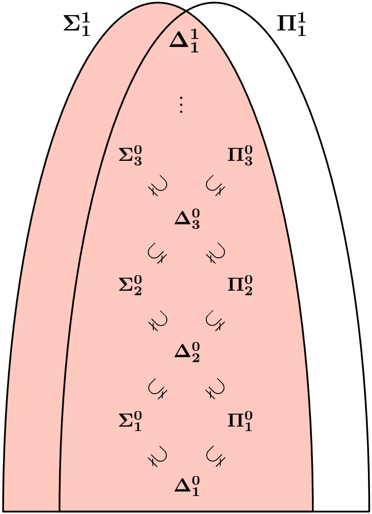
Topology


$\boldsymbol{\Sigma}^1_1$
$\vdots$
$\boldsymbol{\Sigma}^0_3$ $\boldsymbol{\Pi}^0_3$
$\boldsymbol{\Delta}^0_3$
$\boldsymbol{\Sigma}^0_2$ $\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Delta}^0_2$
$\boldsymbol{\Sigma}^0_1$ $\boldsymbol{\Pi}^0_1$
$\boldsymbol{\Delta}^0_1$

Topology

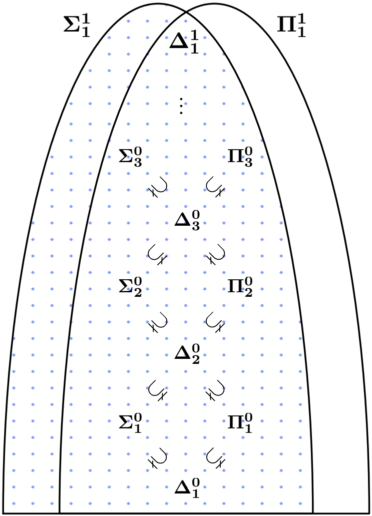

$\mathbf{\Sigma}^1_1$
$\mathbf{\Pi}^1_1$
$\mathbf{\Delta}^1_1$
$\vdots$
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$\mathbf{\Delta}^0_3$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$
$\mathbf{\Delta}^0_1$

Topology



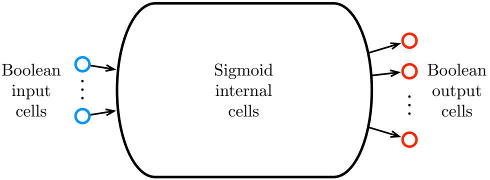
$\boldsymbol{\Sigma}^1_1$
$\boldsymbol{\Pi}^1_1$
$\boldsymbol{\Delta}^1_1$
$\vdots$
$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Pi}^0_3$
$\boldsymbol{\Delta}^0_3$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Delta}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$
$\boldsymbol{\Delta}^0_1$

Topology


$\mathbf{\Sigma}_1^1$
$\mathbf{\Pi}_1^1$
$\mathbf{\Delta}_1^1$
$\vdots$
$\mathbf{\Sigma}_3^0$
$\mathbf{\Pi}_3^0$
$\mathbf{\Delta}_3^0$
$\mathbf{\Sigma}_2^0$
$\mathbf{\Pi}_2^0$
$\mathbf{\Delta}_2^0$
$\mathbf{\Sigma}_1^0$
$\mathbf{\Pi}_1^0$
$\mathbf{\Delta}_1^0$

Deterministic ω-RNNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff $\mathcal{N}(s)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ rejected by $\mathcal{N}$ otherwise.

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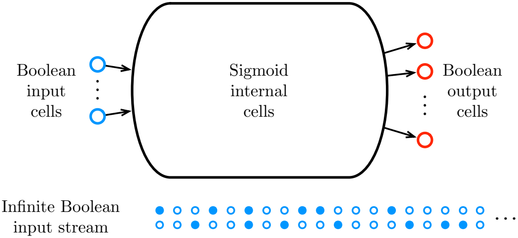

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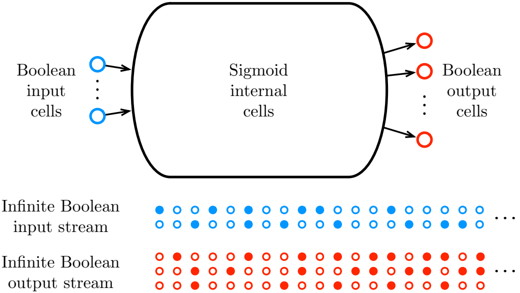

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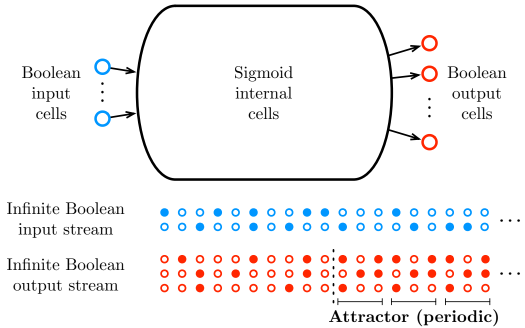

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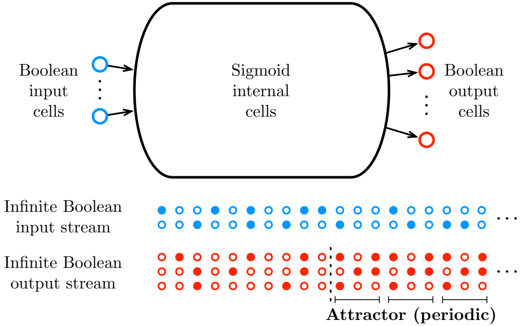

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Deterministic ω-RNNs

We consider six models of deterministic RNNs:

1. static rational RNNs: D-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s

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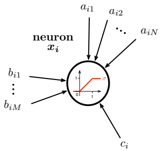
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}\cdot u_j(t) + c_i\right)$$

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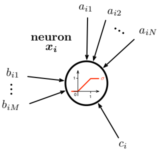
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

Deterministic ω-RNNs

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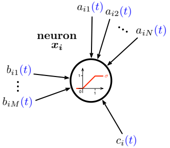
2. static real RNNs: D-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: D-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

Deterministic ω-RNNs

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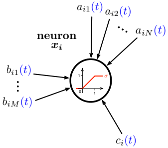

$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

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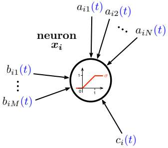
4. bi-valued evolving real RNNs: D-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: D-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

Deterministic ω-RNNs

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5. general evolving rational RNNs: D-Ev-RNN[$\mathbb{Q}$]s
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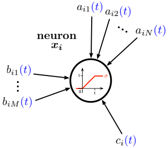


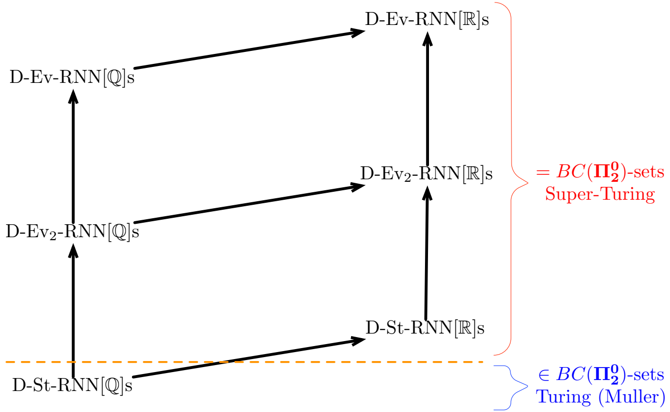
$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

Deterministic ω-RNNs

One has the following relationships between those models:

$$\begin{tikzcd}
 & \text{D-Ev-RNN}[\mathbb{R}]\text{s} \\
\text{D-Ev-RNN}[\mathbb{Q}]\text{s} \arrow[ur] & \\
 & \text{D-Ev}_2\text{-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{D-Ev}_2\text{-RNN}[\mathbb{Q}]\text{s} \arrow[uu] \arrow[ur] & \\
 & \text{D-St-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{D-St-RNN}[\mathbb{Q}]\text{s} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Results


D-Ev-RNN[$\mathbb{R}$]s
D-Ev-RNN[$\mathbb{Q}$]s
D-Ev$_2$-RNN[$\mathbb{R}$]s
D-Ev$_2$-RNN[$\mathbb{Q}$]s
D-St-RNN[$\mathbb{R}$]s
D-St-RNN[$\mathbb{Q}$]s
$= BC(\mathbf{\Pi^0_2})$-sets
Super-Turing
$\in BC(\mathbf{\Pi^0_2})$-sets
Turing (Muller)

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- *L is recognizable by some deterministic Muller TM (and thus $L \in BC(\mathbf{\Pi^0_2})$)*
- *L is recognizable by some D-St-RNN[$\mathbb{Q}$]*

Proof (sketch): Generalization of the classical equivalence between TMs and St-RNN[$\mathbb{Q}$] (Siegelmann & Sontag 95).

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

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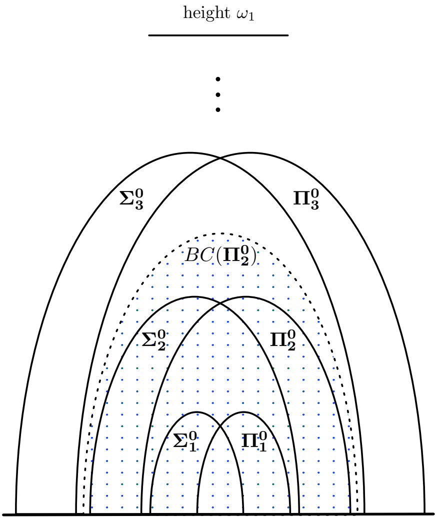
Results


height ω_1
$\vdots$
$\mathbf{\Sigma^0_3}$
$\mathbf{\Pi^0_3}$
$BC(\mathbf{\Pi^0_2})$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

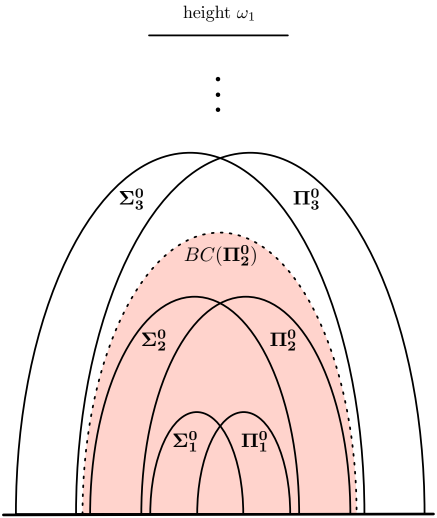
Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\mathbf{\Pi^0_2})$;
- L *is recognizable by some D-St-RNN[$\mathbb{R}$];*
- L *is recognizable by some D-Ev$_2$-RNN[$\mathbb{Q}$];*
- L *is recognizable by some D-Ev-RNN[$\mathbb{Q}$];*
- L *is recognizable by some D-Ev$_2$-RNN[$\mathbb{R}$];*
- L *is recognizable by some D-Ev-RNN[$\mathbb{R}$].*

Results



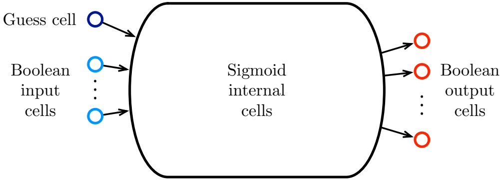
height ω_1
$\vdots$
$\mathbf{\Sigma^0_3}$
$\mathbf{\Pi^0_3}$
$BC(\mathbf{\Pi^0_2})$
$\mathbf{\Sigma^0_2}$
$\mathbf{\Pi^0_2}$
$\mathbf{\Sigma^0_1}$
$\mathbf{\Pi^0_1}$

Results – Summary

DET.	Static	Bi-valued Evolving	General Evolving
$\mathbb{Q}$	D-St-RNN[$\mathbb{Q}$]s $\in BC(\mathbf{\Pi}^0_2)$ Turing (Muller)	D-Ev$_2$-RNN[$\mathbb{Q}$]s $= BC(\mathbf{\Pi}^0_2)$ super-Turing	D-Ev-RNN[$\mathbb{Q}$]s $= BC(\mathbf{\Pi}^0_2)$ super-Turing
$\mathbb{R}$	D-St-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi}^0_2)$ super-Turing	D-Ev$_2$-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi}^0_2)$ super-Turing	D-Ev-RNN[$\mathbb{R}$]s $= BC(\mathbf{\Pi}^0_2)$ super-Turing

Nondeterminism of Type I

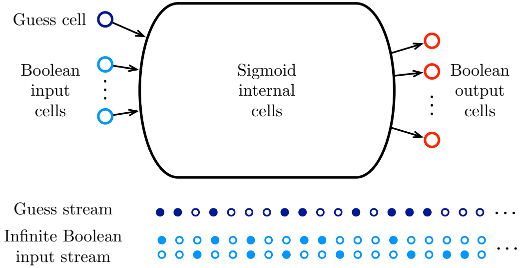
The RNNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s,g)$ enters a meaningful attractor.

Nondeterminism of Type I

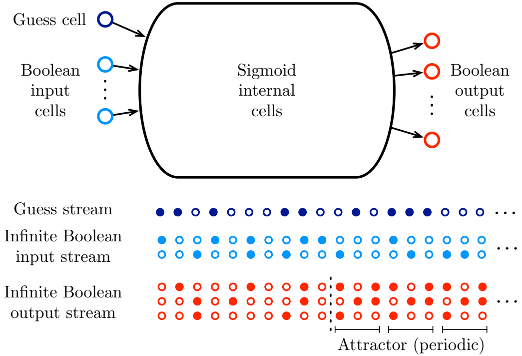
The RNNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

Nondeterminism of Type I

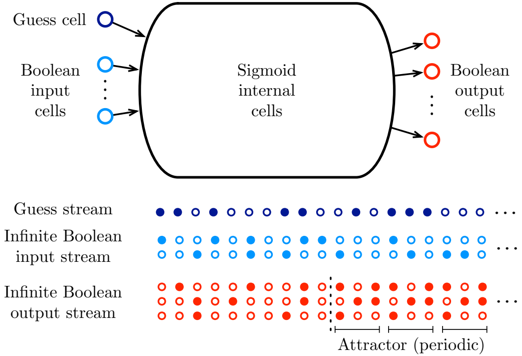
The RNNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

Nondeterminism of Type I

The RNNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

Nondeterminism of Type I

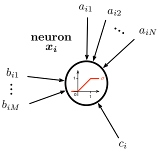
We consider six models of nondeterministic RNNs of type I:

1. static rational RNNs: N-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: N-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: [illegible]
4. bi-valued evolving real RNNs: [illegible]
5. general evolving rational RNNs: [illegible]
6. general evolving real N-RNNs: [illegible]

Nondeterminism of Type I

We consider six models of nondeterministic RNNs of type I:

1. static rational RNNs: N-St-RNN[$\mathbb{Q}$]s
2. static real RNNs: N-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: N-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: N-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}\cdot u_j(t) + c_i\right)$$

Nondeterminism of Type I

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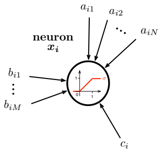
4. bi-valued evolving real RNNs: N-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

Nondeterminism of Type I

We consider six models of nondeterministic RNNs of type I:

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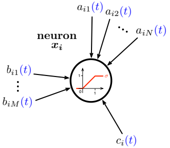
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type I

We consider six models of nondeterministic RNNs of type I:

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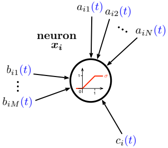
4. bi-valued evolving real RNNs: N-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type I

We consider six models of nondeterministic RNNs of type I:

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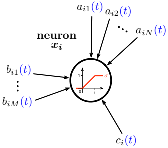
2. static real RNNs: N-St-RNN[$\mathbb{R}$]s
3. bi-valued evolving rational RNNs: N-Ev$_2$-RNN[$\mathbb{Q}$]s
4. bi-valued evolving real RNNs: N-Ev$_2$-RNN[$\mathbb{R}$]s
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

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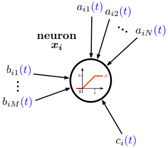
5. general evolving rational RNNs: N-Ev-RNN[$\mathbb{Q}$]s
6. general evolving real N-RNNs: N-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

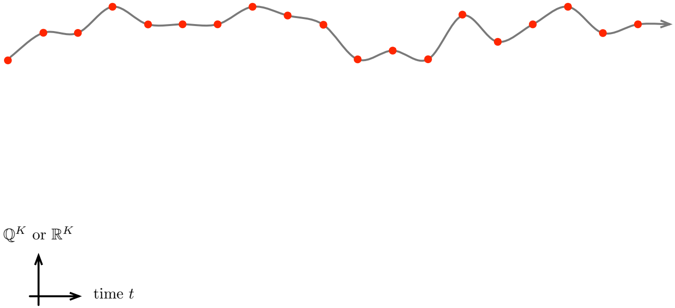
Nondeterminism of Type I

One has the following relationships between those models:

$$
\begin{tikzcd}
 & \text{N-Ev-RNN}[\mathbb{R}]\text{s} \\
\text{N-Ev-RNN}[\mathbb{Q}]\text{s} \arrow[ur] & \\
 & \text{N-Ev}_2\text{-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{N-Ev}_2\text{-RNN}[\mathbb{Q}]\text{s} \arrow[uu] \arrow[ur] & \\
 & \text{N-St-RNN}[\mathbb{R}]\text{s} \arrow[uu] \\
\text{N-St-RNN}[\mathbb{Q}]\text{s} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

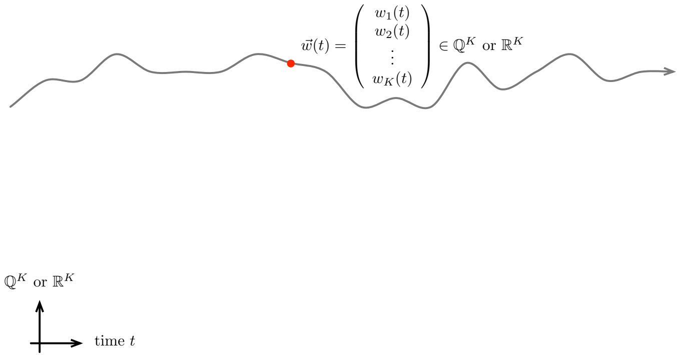
Nondeterminism of Type II

A possible **evolution** $e = \vec{w}(0)\,\vec{w}(1)\,\vec{w}(2)\cdots \in \mathbb{Q}^K$ or $\mathbb{R}^K$



Nondeterminism of Type II

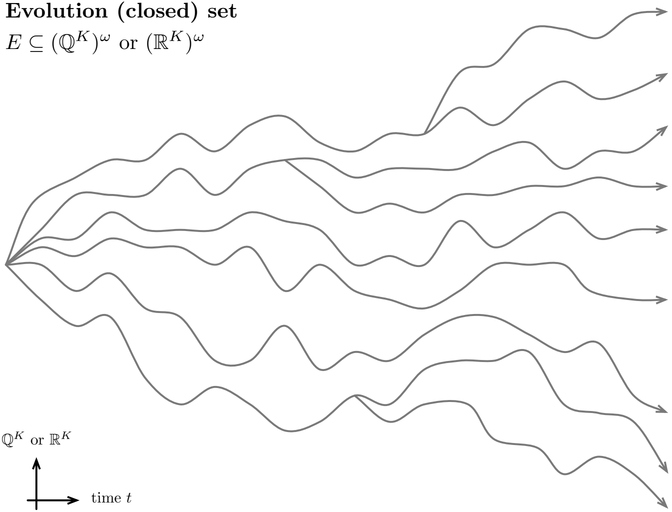
A possible **evolution** $e = \vec{w}(0)\,\vec{w}(1)\,\vec{w}(2) \cdots \in \mathbb{Q}^K$ or $\mathbb{R}^K$



Nondeterminism of Type II

Evolution (closed) set

$E \subseteq (\mathbb{Q}^K)^\omega$ or $(\mathbb{R}^K)^\omega$



Nondeterminism of Type II

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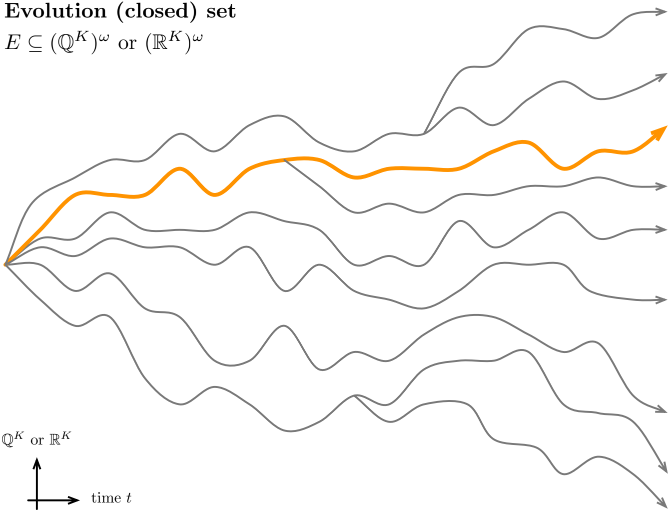
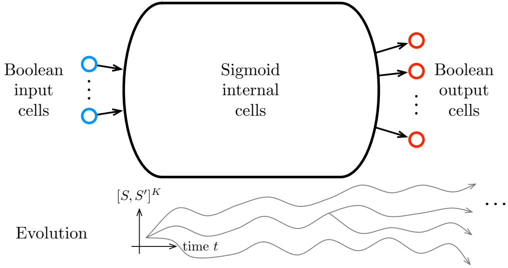


Nondeterminism of Type II

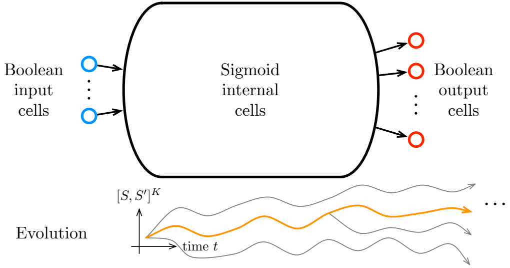
The RNNs are provided with an additional evolution set.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some evolution $e \in E$ s.t. $\mathcal{N}(s, e)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ is rejected by $\mathcal{N}$

Nondeterminism of Type II

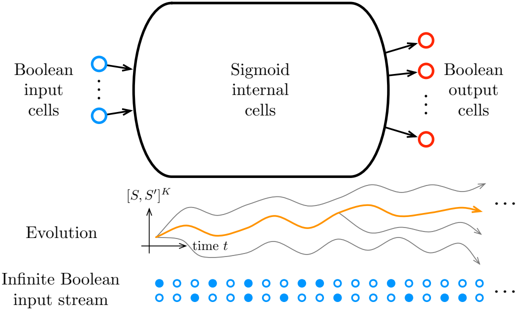
The RNNs are provided with an additional evolution set.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some evolution $e \in E$ s.t. $\mathcal{N}(s, e)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

Nondeterminism of Type II

The RNNs are provided with an additional evolution set.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some evolution $e \in E$ s.t. $\mathcal{N}(s, e)$ enters a meaningful attractor.
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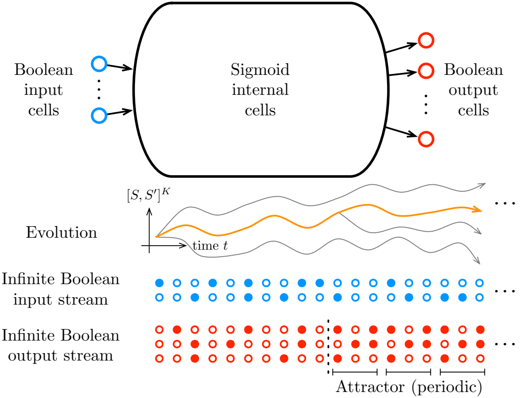

- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some evolution $e \in E$ s.t. $\mathcal{N}(s, e)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

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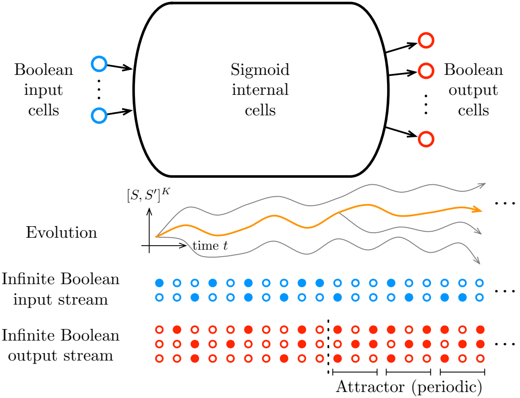


- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some evolution $e \in E$ s.t. $\mathcal{N}(s, e)$ enters a meaningful attractor.
- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ otherwise.

Nondeterminism of Type II

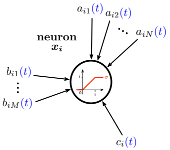
We consider four models of nondeterministic RNNs of type II:

1. bi-valued evolving rational RNNs: $\tilde{N}$-Ev_2-$RNN[\mathbb{Q}]$s
2. bi-valued evolving real RNNs: $\tilde{N}$-Ev_2-$RNN[\mathbb{R}]$s
3. general evolving rational RNNs: $\tilde{N}$-Ev-$RNN[\mathbb{Q}]$s
4. general evolving real N-RNNs: $\tilde{N}$-Ev-$RNN[\mathbb{R}]$s

Nondeterminism of Type II

We consider four models of nondeterministic RNNs of type II:

1. bi-valued evolving rational RNNs: $\tilde{\text{N}}$-Ev_2-$\text{RNN}[\mathbb{Q}]$s
2. bi-valued evolving real RNNs: $\tilde{\text{N}}$-Ev_2-$\text{RNN}[\mathbb{R}]$s
3. general evolving rational RNNs: $\tilde{\text{N}}$-Ev-$\text{RNN}[\mathbb{Q}]$s
4. general evolving real N-RNNs: $\tilde{\text{N}}$-Ev-$\text{RNN}[\mathbb{R}]$s

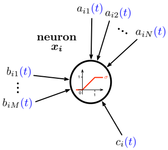


$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type II

We consider four models of nondeterministic RNNs of type II:

1. bi-valued evolving rational RNNs: $\tilde{\text{N}}$-Ev$_2$-RNN[$\mathbb{Q}$]s
2. bi-valued evolving real RNNs: $\tilde{\text{N}}$-Ev$_2$-RNN[$\mathbb{R}$]s
3. general evolving rational RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{Q}$]s
4. general evolving real N-RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type II

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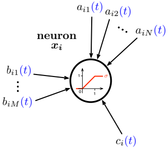
3. general evolving rational RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{Q}$]s
4. general evolving real N-RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type II

We consider four models of nondeterministic RNNs of type II:

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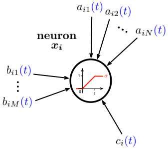
3. general evolving rational RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{Q}$]s
4. general evolving real N-RNNs: $\tilde{\text{N}}$-Ev-RNN[$\mathbb{R}$]s



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t) \cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t) \cdot u_j(t) + c_i(t)\right)$$

Nondeterminism of Type II

One has the following relationships between those models:

$$
\begin{tikzcd}
\tilde{\mathrm{N}}\text{-Ev-RNN}[\mathbb{Q}]\text{s} \arrow[r, dashed] & \tilde{\mathrm{N}}\text{-Ev-RNN}[\mathbb{R}]\text{s} \\
\tilde{\mathrm{N}}\text{-Ev}_2\text{-RNN}[\mathbb{Q}]\text{s} \arrow[u, dashed] \arrow[r, dashed] & \tilde{\mathrm{N}}\text{-Ev}_2\text{-RNN}[\mathbb{R}]\text{s} \arrow[u, dashed]
\end{tikzcd}
$$

Nondeterminism of Types I and II

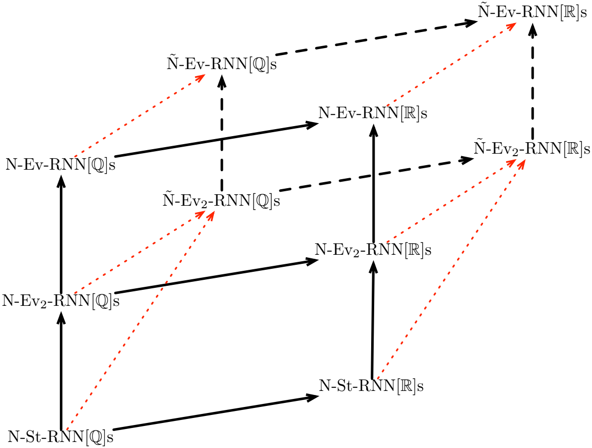
Lemma

The nondeterminism of type I is a particular case of that of type II.

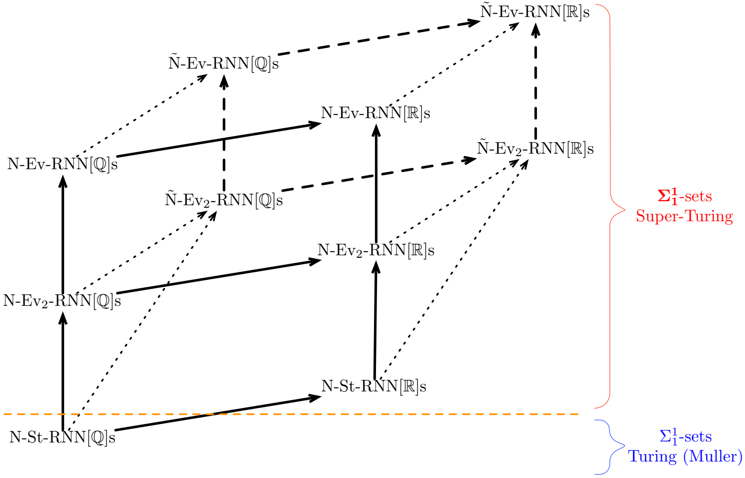
Proof: simply build an evolution set that takes into account all possible guess streams.

Nondeterminism of Types I and II

One has the following relationships between all the models:



Results


$\tilde{\text{N}}$-Ev-RNN[$\mathbb{R}$]s
$\tilde{\text{N}}$-Ev-RNN[$\mathbb{Q}$]s
N-Ev-RNN[$\mathbb{R}$]s
$\tilde{\text{N}}$-Ev$_2$-RNN[$\mathbb{R}$]s
N-Ev-RNN[$\mathbb{Q}$]s
$\tilde{\text{N}}$-Ev$_2$-RNN[$\mathbb{Q}$]s
N-Ev$_2$-RNN[$\mathbb{R}$]s
N-Ev$_2$-RNN[$\mathbb{Q}$]s
N-St-RNN[$\mathbb{R}$]s
N-St-RNN[$\mathbb{Q}$]s
$\boldsymbol{\Sigma^1_1}$-sets
Super-Turing
Σ^1_1-sets
Turing (Muller)

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \Sigma^1_1$
- L *is recognizable by some nondeterministic Muller TM*
- L *is recognizable by some N-St-RNN[$\mathbb{Q}$]*

Proof: Generalization of the classical equivalence between TMs and St-RNN[$\mathbb{Q}$] (Siegelmann & Sontag 95).

Results

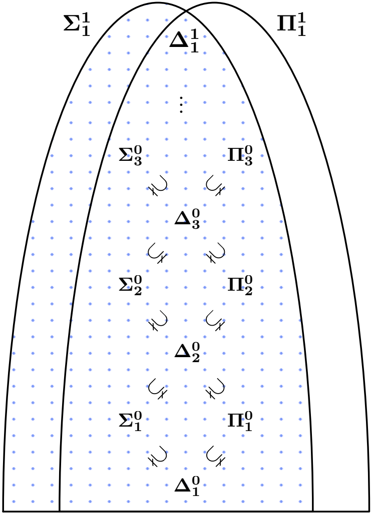
Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \Sigma^1_1$
- L *is recognizable by some nondeterministic Muller TM*
- L *is recognizable by some N-St-RNN[$\mathbb{Q}$]*

Proof: Generalization of the classical equivalence between TMs and St-RNN[$\mathbb{Q}$] (Siegelmann & Sontag 95).

Results


$\mathbf{\Sigma}^1_1$
$\mathbf{\Pi}^1_1$
$\mathbf{\Delta}^1_1$
$\vdots$
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$\mathbf{\Delta}^0_3$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$
$\mathbf{\Delta}^0_1$

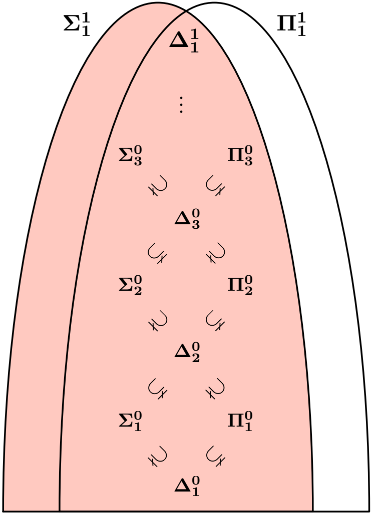
Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

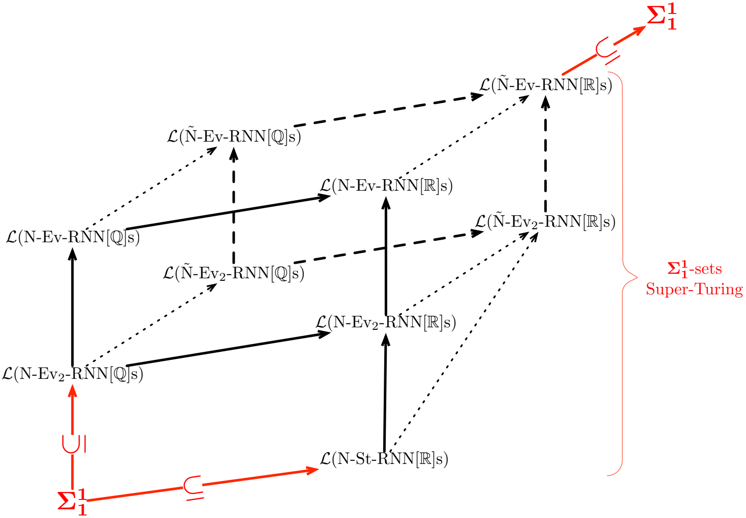
- $L \in \mathbf{\Sigma}^1_1$;
- L *is recognizable by some N-St-RNN[$\mathbb{R}$];*
- L *is recognizable by some N-Ev$_2$-RNN[$\mathbb{Q}$];*
- L *is recognizable by some Ñ-Ev$_2$-RNN[$\mathbb{Q}$];*
- L *is recognizable by some N-Ev-RNN[$\mathbb{Q}$];*
- L *is recognizable by some Ñ-Ev-RNN[$\mathbb{Q}$];*
- L *is recognizable by some N-Ev$_2$-RNN[$\mathbb{R}$];*
- L *is recognizable by some Ñ-Ev$_2$-RNN[$\mathbb{R}$];*
- L *is recognizable by some N-Ev-RNN[$\mathbb{R}$].*
- L *is recognizable by some Ñ-Ev-RNN[$\mathbb{R}$].*

Results



$\mathbf{\Sigma}^1_1$
$\mathbf{\Pi}^1_1$
$\mathbf{\Delta}^1_1$
$\vdots$
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$\mathbf{\Delta}^0_3$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$
$\mathbf{\Delta}^0_1$

Results


Σ^1_1
$\subsetneq$
$\mathcal{L}$(Ñ-Ev-RNN[$\mathbb{R}$]s)
$\mathcal{L}$(Ñ-Ev-RNN[$\mathbb{Q}$]s)
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$\mathcal{L}$(N-Ev-RNN[$\mathbb{Q}$]s)
$\mathcal{L}$(Ñ-Ev$_2$-RNN[$\mathbb{Q}$]s)
Σ^1_1-sets
Super-Turing
$\mathcal{L}$(N-Ev$_2$-RNN[$\mathbb{R}$]s)
$\mathcal{L}$(N-Ev$_2$-RNN[$\mathbb{Q}$]s)
$\subsetneq$
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Σ^1_1
$\subsetneq$

Results

Proposition

Let $L \in \mathbf{\Sigma}^1_1$. Then L is recognizable by some N-Ev$_2$-RNN[$\mathbb{Q}$] or by some N-St-RNN[$\mathbb{R}$].

Proof (sketch):

- Since $L \in \Sigma^1_1$, there exists some Π^0_2 set $X \subseteq (\mathbb{B}^M)^\omega \times \{0,1\}^\omega$ such that $L = \pi_1(X)$.

$$X = \bigcap_{i \geq 0} \bigcup_{j \geq 0} \left(p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega \right)$$

- X can be recursively encoded into some $r_X \in \mathbb{R}$.

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Proof (sketch):

- Since $L \in \mathbf{\Sigma}^1_1$, there exists some $\mathbf{\Pi}^0_2$ set $X \subseteq (\mathbb{B}^M)^\omega \times \{0,1\}^\omega$ such that $L = \pi_1(X)$.

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Results

Algorithm 1 Infinite procedure

Require: Input stream $s = \vec{u}(0)\vec{u}(1)\vec{u}(2)\cdots \in (\mathbb{B}^M)^\omega$,
guess stream $g = g(0)g(1)g(2)\cdots \in \mathbb{B}^\omega$, and real number r_X.

store each $\vec{u}(t) \in \mathbb{B}^M$ and $g(t) \in \{0,1\}$ as they arrive
$i \leftarrow 0$, $j \leftarrow 0$
loop
  decode $(p_{i,j}, q_{i,j})$ from r_X // recursive procedure if r_X is given
  if $p_{i,j} \subseteq s[0{:}c]$ and $q_{i,j} \subseteq g[0{:}c]$ **then**   // $(s,g) \in p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega$
    return 1 // $\exists\ j$ s.t. $(s,g) \in p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega$
    $i \leftarrow i+1$, $j \leftarrow 0$ // test if $(s,g) \in p_{i+1,0} \cdot (\mathbb{B}^M)^\omega \times q_{i+1,0} \cdot \{0,1\}^\omega$
  else   // $(s,g) \notin p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega$
    return 0 // $\neg\exists j' \leq j$ s.t. $(s,g) \in p_{i,j'} \cdot (\mathbb{B}^M)^\omega \times q_{i,j'} \cdot \{0,1\}^\omega$
    $i \leftarrow i$, $j \leftarrow j+1$ // test if $(s,g) \in p_{i,j+1} \cdot (\mathbb{B}^M)^\omega \times q_{i,j+1} \cdot \{0,1\}^\omega$
  end if
end loop

Algo returns ∞-many 1's iff $(s,g) \in \bigcap_i \bigcup_j (p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega) = X$.

Results

Algorithm 2 Infinite procedure

Require: Input stream $s = \vec{u}(0)\vec{u}(1)\vec{u}(2)\cdots \in (\mathbb{B}^M)^\omega$,
guess stream $g = g(0)g(1)g(2)\cdots \in \mathbb{B}^\omega$, and real number r_X.

```
store each u⃗(t) ∈ 𝔹^M and g(t) ∈ {0,1} as they arrive
i ← 0, j ← 0
loop
    decode (p_{i,j}, q_{i,j}) from r_X             // recursive procedure if r_X is given
    if p_{i,j} ⊆ s[0:c] and q_{i,j} ⊆ g[0:c] then  // (s,g) ∈ p_{i,j}·(𝔹^M)^ω × q_{i,j}·{0,1}^ω
        return 1                                   // ∃ j s.t. (s,g) ∈ p_{i,j}·(𝔹^M)^ω × q_{i,j}·{0,1}^ω
        i ← i+1, j ← 0                             // test if (s,g) ∈ p_{i+1,0}·(𝔹^M)^ω × q_{i+1,0}·{0,1}^ω
    else                                           // (s,g) ∉ p_{i,j}·(𝔹^M)^ω × q_{i,j}·{0,1}^ω
        return 0                                   // ¬∃j' ≤ j s.t. (s,g) ∈ p_{i,j'}·(𝔹^M)^ω × q_{i,j'}·{0,1}^ω
        i ← i, j ← j+1                             // test if (s,g) ∈ p_{i,j+1}·(𝔹^M)^ω × q_{i,j+1}·{0,1}^ω
    end if
end loop
```

Algo returns ∞-many 1's iff $(s,g) \in \bigcap_i \bigcup_j (p_{i,j} \cdot (\mathbb{B}^M)^\omega \times q_{i,j} \cdot \{0,1\}^\omega) = X$.

Results

- The algo can be simulated by some N-St-RNN[$\mathbb{R}$] $\mathcal{N}$, i.e.,

  algo returns infinitely many 1's on (s, g)
  iff
  $\mathcal{N}(s, g)$ visits a meaningful attractor.

- $s \in L(\mathcal{N})$
  - iff $\exists\, g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ visits a meaningful attractor
  - iff $\exists\, g \in \mathbb{B}^\omega$ s.t. the algo returns infinitely many 1's on (s, g)
  - iff $\exists\, g \in \mathbb{B}^\omega$ s.t. $(s, g) \in X$
  - iff $s \in \pi_1(X) = L$.

- Therefore, $L(\mathcal{N}) = L$. $\square$

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  - iff $s \in \pi_1(X) = L$.
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Results

Proposition

Let $\mathcal{N}$ be some $\tilde{\mathrm{N}}$-Ev-RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma}^1_1$.

Proof (sketch):

- The function $f_{\mathcal{N}} : (\mathbb{B}^M)^\omega \times E \to (\mathbb{B}^P)^\omega$ associated with the dynamics of $\mathcal{N}$ is of Baire class 1 (preimage of a $\mathbf{\Sigma}^0_1$ is a $\mathbf{\Sigma}^0_2$).
- Accordingly, the ω-language $L(\mathcal{N})$ can be expressed as the first projection of a finite Boolean combination of $\mathbf{\Sigma}^0_3$ and $\mathbf{\Pi}^0_3$ sets (i.e., of a Borel set) of the Polish space $(\mathbb{B}^M)^\omega \times E$.
- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma}^1_1$. □

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Let $\mathcal{N}$ be some Ñ-Ev-RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma}^1_1$.

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Proposition

Let $\mathcal{N}$ be some Ñ-Ev-RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma}_1^1$.

Proof (sketch):

- The function $f_{\mathcal{N}} : (\mathbb{B}^M)^\omega \times E \to (\mathbb{B}^P)^\omega$ associated with the dynamics of $\mathcal{N}$ is of Baire class 1 (preimage of a $\mathbf{\Sigma}_1^0$ is a $\mathbf{\Sigma}_2^0$).
- Accordingly, the ω-language $L(\mathcal{N})$ can be expressed as the first projection of a finite Boolean combination of $\mathbf{\Sigma}_3^0$ and $\mathbf{\Pi}_3^0$ sets (i.e., of a Borel set) of the Polish space $(\mathbb{B}^M)^\omega \times E$.
- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma}_1^1$. $\square$

Results – Summary

NONDET.	Static	Bi-valued Evolving	General Evolving
$\mathbb{Q}$	N-St-RNN[$\mathbb{Q}$]s	N-Ev$_2$-RNN[$\mathbb{Q}$]s	N-Ev-RNN[$\mathbb{Q}$]s
	–	Ñ-Ev$_2$-RNN[$\mathbb{Q}$]s	Ñ-Ev-RNN[$\mathbb{Q}$]s
	$= \Sigma^1_1$ (lightface)	$= \boldsymbol{\Sigma}^1_1$ (boldface)	$= \boldsymbol{\Sigma}^1_1$ (boldface)
	Turing (Muller)	super-Turing	super-Turing
$\mathbb{R}$	N-St-RNN[$\mathbb{R}$]s	N-Ev$_2$-RNN[$\mathbb{R}$]s	N-Ev-RNN[$\mathbb{R}$]s
	–	Ñ-Ev$_2$-RNN[$\mathbb{R}$]s	Ñ-Ev-RNN[$\mathbb{R}$]s
	$= \boldsymbol{\Sigma}^1_1$ (boldface)	$= \boldsymbol{\Sigma}^1_1$ (boldface)	$= \boldsymbol{\Sigma}^1_1$ (boldface)
	super-Turing	super-Turing	super-Turing

Conclusion

- We provided a characterization of the expressive power of recurrent neural networks in terms of their attractor dynamics.
- In general, the super-Turing computational capabilities of neural models raises the question of *hypercomputation*.
- Current physical theories are consistent with the possibility of hypercomputational systems (quantum, relativistic, etc.). No such systems are currently feasible or harnessable.
- Philosophical and scientific literature about hypercomputation is however flourishing.

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