The Algebraic Counterpart of the Wagner Hierarchy

PhD Thesis

Jérémie Cabessa

University of Lausanne - ISI
University of Paris 7 - LIAFA

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Outline

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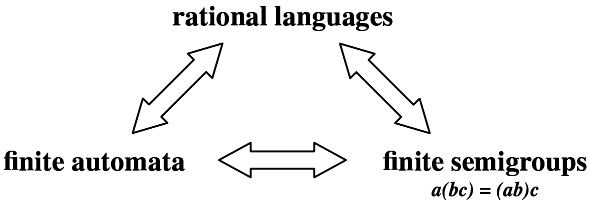
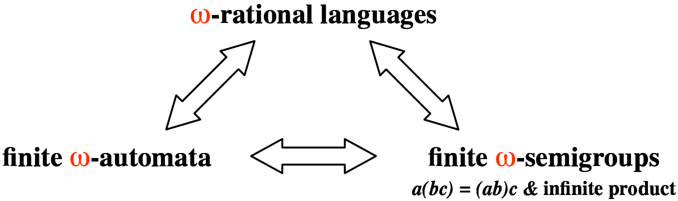
Classifying ω-rational languages

Automata
The Wagner Hierarchy
[Wag79]

Descriptive set theory
The Wadge hierarchy
of ω-rational languages
[Sel98]

Algebra
[CarPer96,99], [DupRis06]
The FSG-hierarchy


rational languages
finite automata
finite semigroups
$a(bc) = (ab)c$


ω-rational languages
finite ω-automata
finite ω-semigroups
a(bc) = (ab)c & infinite product

An ω-semigroup is a semigroup equipped with a suitable infinite product.

Definition (ω-semigroup)

An ω-semigroup S is a pair (S_+, S_ω), where S_+ is a semigroup, S_ω is a set, and equipped with

- an associative mixed product : $S_+ \times S_\omega \longrightarrow S_\omega$
- a surjective infinite product $\pi_S : S_+^\omega \longrightarrow S_\omega$ satisfying

$$\pi_S(s_0 s_1 \cdots s_{k_1 - 1}, s_{k_1} \cdots s_{k_2 - 1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$

$$s \pi_S(s_0, s_1, s_2, \ldots) = \pi_S(s, s_0, s_1, s_2, \ldots).$$

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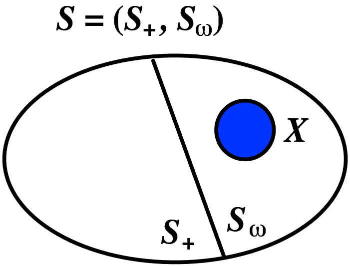
Notation: $\pi_S(x, x, x, x, \ldots)$ will be written x^ω.

A *pointed ω-semigroup* is a pair (S, X), where

- $S = (S_+, S_\omega)$ is an ω-semigroup,
- $X \subseteq S_\omega$.



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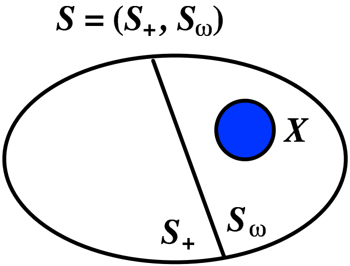

Finite pointed ω-semigroups are the algebraic counterparts of Büchi automata.

Theorem

An ω-language is recognizable by a finite pointed ω-semigroup iff it is recognizable by a finite Büchi automaton (iff it is ω-rational).

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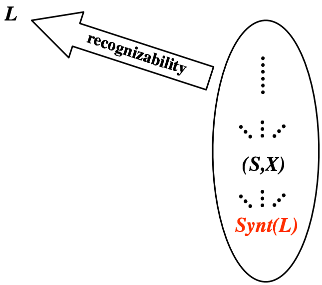
Among all finite pointed ω-semigroups recognizing a given ω-rational language L, there exists a minimal one, **the syntactic ω-semigroup** of L, denoted by $\boldsymbol{Synt(L)}$.



Example

Consider the language $K = ((a+b)^*a)^\omega$. Then $Synt(K) = (S, X)$, where

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0\cdot 0 = 0 \quad 0\cdot 1 = 0 \quad 1\cdot 0 = 0 \quad 1\cdot 1 = 1$$

$$00^\omega = 0^\omega \quad 10^\omega = 0^\omega \quad 01^\omega = 1^\omega \quad 11^\omega = 1^\omega$$

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Example

Consider the language $L = (a\{b,c\}^* \cup \{b\})^\omega$. Then $Synt(L) = (T, Y)$, where

- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

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\begin{array}{llll}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
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- $Y = \{a^\omega\}$.

Definition (linked pair)

Let S_+ be a semigroup. A pair $(s, e) \in S_+^2$ is called a *linked pair* if

- $se = s$
- e is idempotent (i.e. $e^2 = e$)

s is called the *prefix*, and e the *idempotent*.

We aim to classify finite pointed ω-semigroups. We adopt a hierarchical game approach.

Main idea: **we transpose Wadge games on ω-semigroups...** We define the $\mathbb{SG}$-*game*, and the corresponding reduction relation: $(S, X) \leq_{SG} (T, Y)$ iff Player II has a w. s. in $\mathbb{SG}((S, X), (T, Y))$.

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Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let also $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.

The infinite two-player game $\mathbb{SG}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as follows:

- Player I plays elements from S_+,
- Player II plays elements from T_+,
- players I and II play alternately, Player I begins,
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& & \searrow & \nearrow & & \searrow & & & \\
\mathrm{II}: & & - & & & t_0 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (t_0, t_1, t_2, \cdots)
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- Player II wins the game $\mathbb{SG}(X, Y)$ iff

$$\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y.$$

Definition ($\mathbb{SG}$-reduction)

We write $X \leq_{SG} Y$ iff Player II has a winning strategy in $\mathbb{SG}(X, Y)$. And as usual $X \equiv_{SG} Y$ iff $X \leq_{SG} Y \leq_{SG} X$.

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Example

Let $S = (\{0,1\}, \{0^{\rightarrow}0, 0^{\rightarrow}1, 1^{\rightarrow}1\})$ be the ω-semigroup defined by the operations:

$0 \cdot 0 = 0$ $0 \cdot 1 = 0$ $1 \cdot 0 = 0$ $1 \cdot 1 = 1$

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- $\pi_S(\alpha) = 0^{\rightarrow}0$ iff α contains infinitely many 0's,
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Then $\{1^{\rightarrow}1\} \leq_{SG} \{0^{\rightarrow}0\}$.

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Example (continued)

We give a w. s. for Player II in $\mathbb{SG}\left(\{1^{\to}1\}, \{0^{\to}0\}\right)$:

Player I	1	1	1	$\cdots$	$\cdots$	$1^{\to}1 \in \{1^{\to}1\}$
Player II	0	0	0	$\cdots$	$\cdots$	$0^{\to}0 \in \{0^{\to}0\}$

Therefore Player II wins.

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Player I	1	1	0	$\cdots\ \cdots$	$0^{\rightarrow}0$ or $0^{\rightarrow}1 \not\in \{1^{\rightarrow}1\}$
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Definition ($\mathbb{FSG}$-hierarchy)

The collection of finite pointed ω-semigroups ordered by the $\leq_{SG}$-relation is called *the* $\mathbb{FSG}$-*hierarchy*.

Proposition ($\mathbb{SG}$-Borel Determinacy)

Let (S, X) and (T, Y) be two finite pointed ω-semigroups. Then the game $\mathbb{SG}(X, Y)$ is determined.

Proof.

A consequence of Borel Wadge determinacy. A given player has a w. s. in $\mathbb{SG}(X, Y)$ iff this same player has a w. s. in the $\mathbb{W}(\pi_S^{-1}(X), \pi_T^{-1}(Y))$. □

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Corollary

- *The $\leq_{SG}$-antichains have length at most two.*
- *If X and Y are incomparable, then $X \equiv_{SG} Y^c$.*
- *The partial ordering $\leq_{SG}$ is wellfounded on finite pointed ω-semigroups.*

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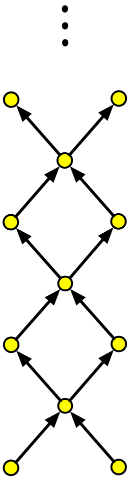
The $\mathbb{FSG}$-hierarchy



Theorem

The $\mathbb{FSG}$-hierarchy and the Wagner hierarchy are isomorphic.

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The mapping which associates every ω-rational language with its syntactic image is an isomorphism between the Wagner and the $\mathbb{FSG}$-hierarchies. $\square$

Corollary

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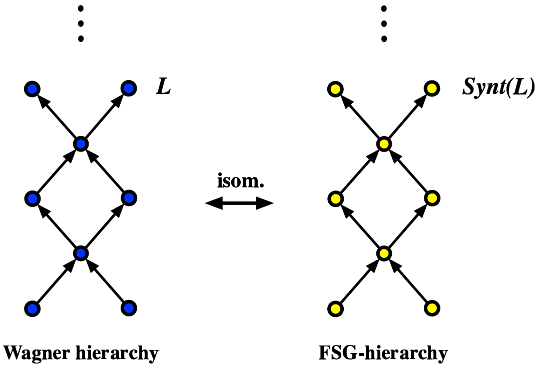
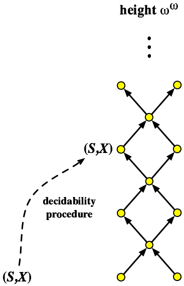
An ω-rational language and its syntactic image have the same degrees.



Decidability procedure of the $\mathbb{FSG}$-hierarchy:



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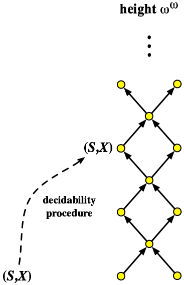

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Linked pairs play a crucial role!

- prefixes: stable positions
- idempotents: waiting moves

Let (s, e) be a linked pair (i.e. $se = s$ and $e^2 = e$)

SG-game	**I**	$a_0 \quad a_1 \quad \ldots \quad a_n$
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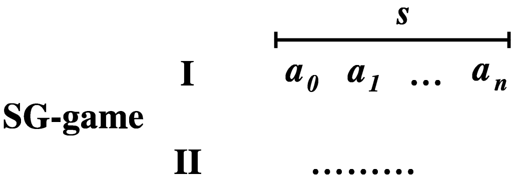

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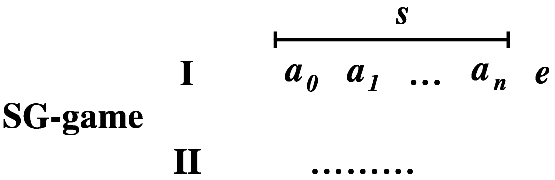

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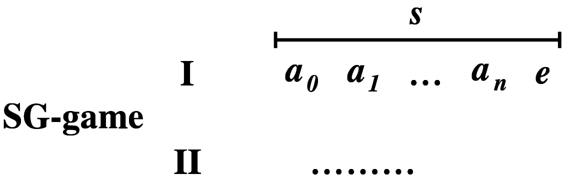

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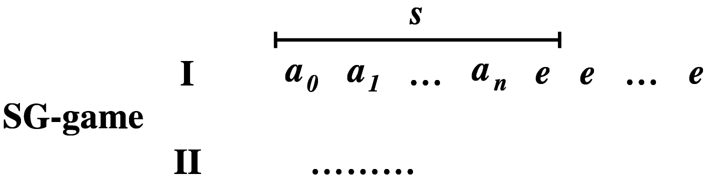

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$$\overbrace{a_0 \quad a_1 \quad \ldots \quad a_n}^{s} \quad e \quad e \quad \ldots \quad e$$

SG-game

I $\quad a_0 \quad a_1 \quad \ldots \quad a_n \quad e \quad e \quad \ldots \quad e$

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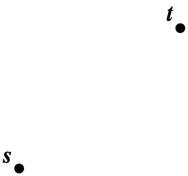
Graph representation of (S, X) (part 1):

Accessibility relation between stable positions



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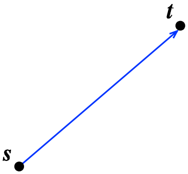

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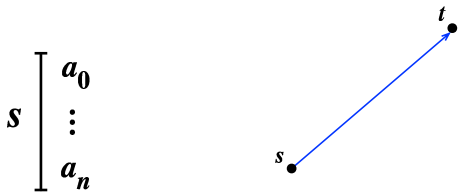

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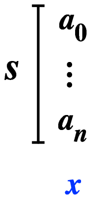

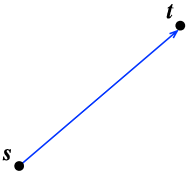

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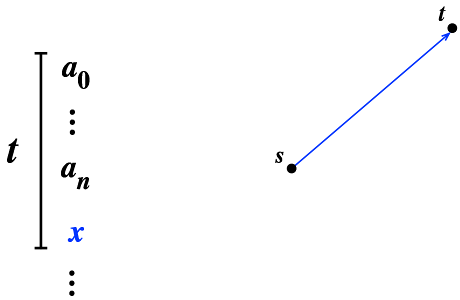

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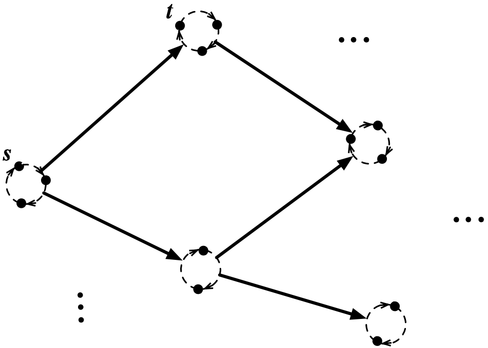
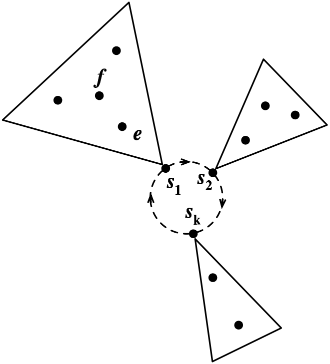


Graph representation of (S, X) (part 2):

Absorption relation and signature of waiting moves



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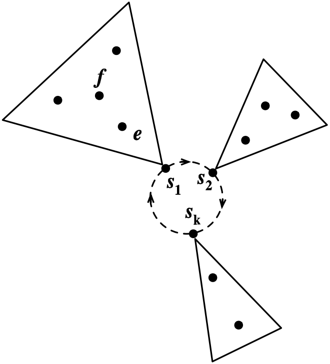
$$s_1 \left[\begin{array}{c} a_0 \\ \vdots \\ a_n \end{array} \right.$$



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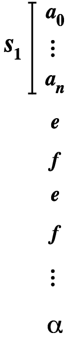

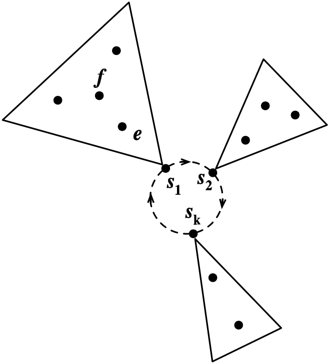

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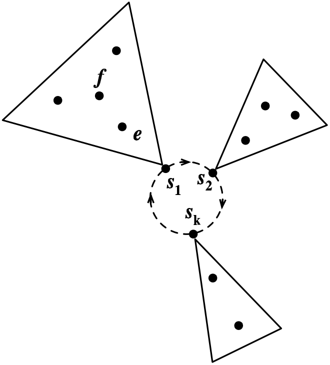
$$\begin{array}{c} s_1\left[\begin{array}{c} a_0\\ \vdots\\ a_n\end{array}\right. \\ e\\ f\\ e\\ f\\ \vdots\\ \alpha\end{array} \qquad \left.\begin{array}{c} a_0\\ \vdots\\ a_n\end{array}\right] s_1$$



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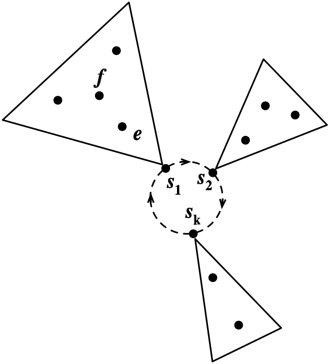
$$
\begin{array}{ccccc}
 & \left[\begin{array}{c} a_0 \\ \vdots \\ a_n \end{array} \right. & & \left. \begin{array}{c} a_0 \\ \vdots \\ a_n \end{array} \right] & \\
s_1 & & & & s_1 \\
 & e & & f & \\
 & f & & f & \\
 & e & & f & \\
 & f & & f & \\
 & \vdots & & \vdots & \\
 & \alpha & & \beta &
\end{array}
$$



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$$
\begin{array}{ccc}
 & a_0 & a_0 & \\
s_1 & \vdots & \vdots & s_1 \\
 & a_n & a_n & \\
 & e & f & \\
 & f & f & \\
 & e & f & \\
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 & \vdots & \vdots & \\
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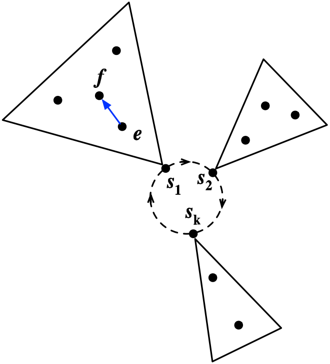
\end{array}
$$



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\begin{array}{ccc}
s_1 & \left[\begin{array}{c} a_0 \\ \vdots \\ a_n \end{array}\right. & \\
 & e & \\
 & f & \\
 & e & \\
 & f & \\
 & \vdots &
\end{array}
\qquad
\begin{array}{ccc}
 & \left.\begin{array}{c} a_0 \\ \vdots \\ a_n \end{array}\right] & s_1 \\
 & f & \\
 & f & \\
 & f & \\
 & f & \\
 & \vdots &
\end{array}
$$

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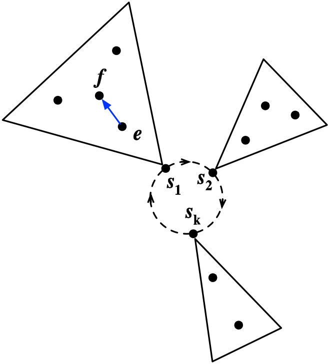

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\cancel{e} & f \\
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\vdots & \vdots
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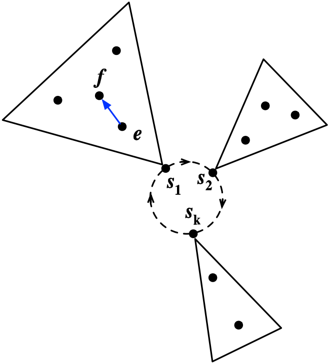

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$$\begin{array}{cccc} & a_0 & a_0 & \\ s_1 & \vdots & \vdots & s_1 \\ & a_n & a_n & \\ & \cancel{e} & f & \\ & f & f & \\ & \cancel{e} & f & \\ & f & f & \\ & \vdots & \vdots & \end{array}$$

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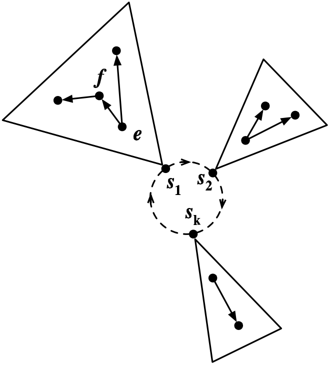

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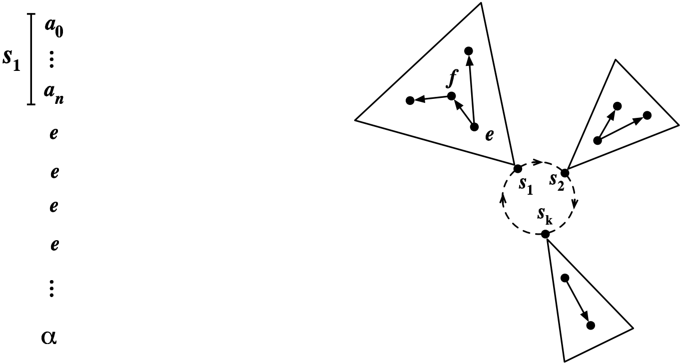

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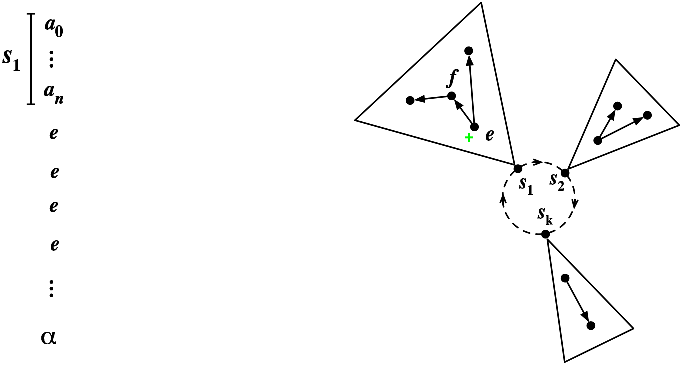


Graph representation of (S, X) (part 2):
Absorption relation and signature of waiting moves

$$s_1 \left[\begin{array}{l} a_0 \\ \vdots \\ a_n \end{array} \right.$$

e

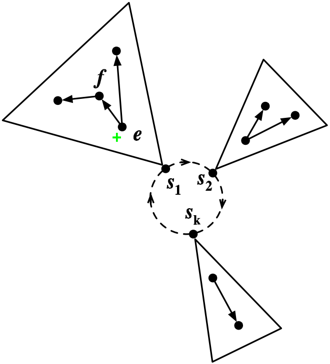
e

e

e

$\vdots$

$\pi_S(\alpha) \in X$



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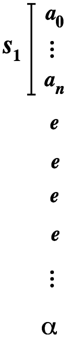

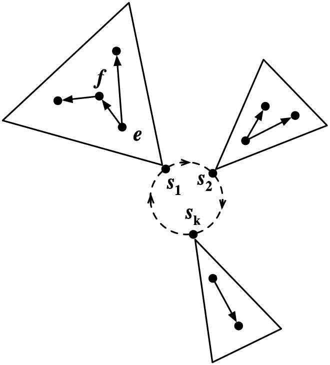

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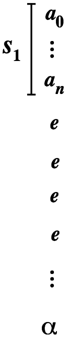

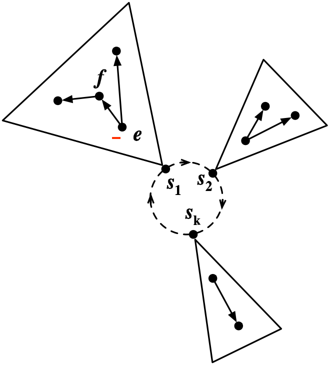

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e

e

e

e

$\vdots$

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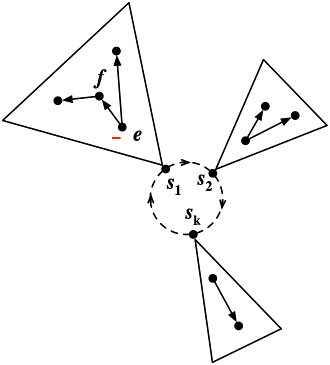

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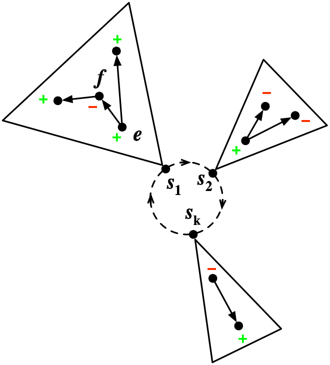

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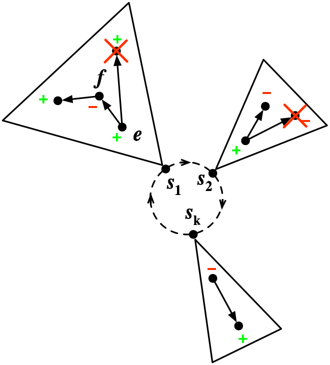

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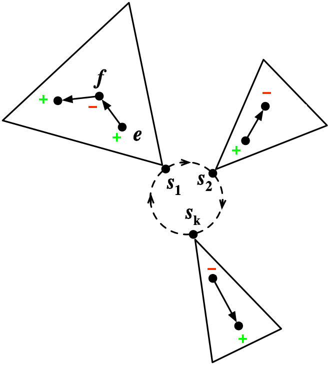

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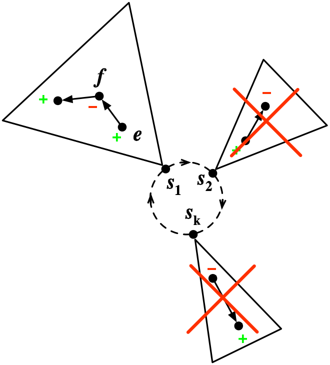

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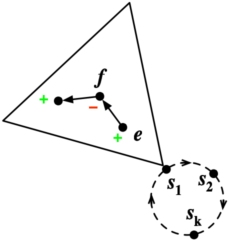
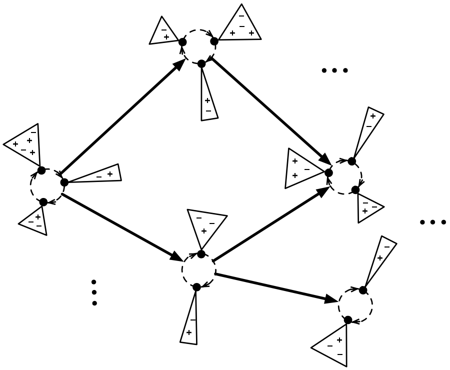


Graph representation of (S, X) (part 3):
Combining all the information



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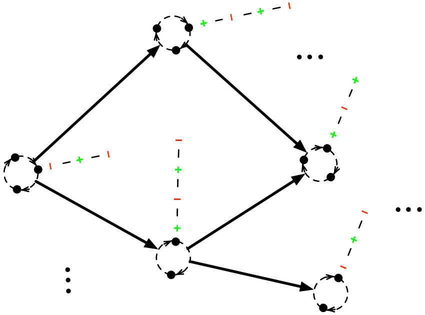

Decidability procedure:

- Analogy: chains of idempotents and chains in Muller automata.
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Example

Consider the syntactic pointed ω-semigroup of $K = ((a+b)^*a)^\omega$:

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by the relations

$$0 \cdot 0 = 0 \quad 0 \cdot 1 = 0 \quad 1 \cdot 0 = 0 \quad 1 \cdot 1 = 1$$

$$00^\omega = 0^\omega \quad 10^\omega = 0^\omega \quad 01^\omega = 1^\omega \quad 11^\omega = 1^\omega$$

- $X = \{0^\omega\}$.

Example (continued)

The linked pairs are: $(0,0)$, $(0,1)$, and $(1,1)$.

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by

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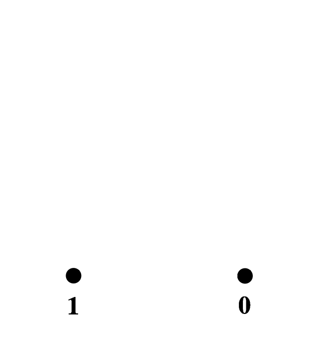

Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

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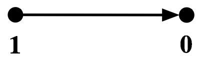

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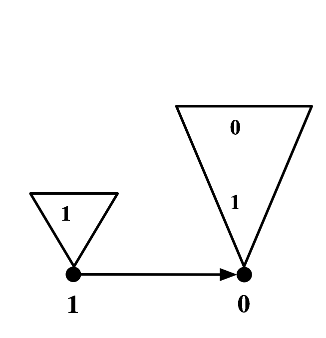

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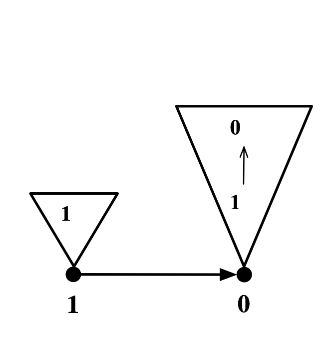

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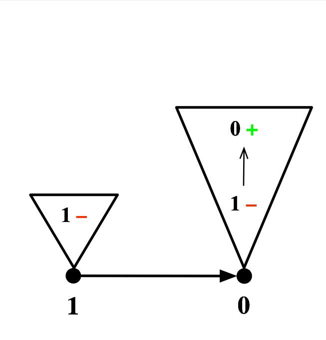
  $$\begin{array}{ll} 0 \cdot 0 = 0 & 0 \cdot 1 = 0 \\ 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega \\ 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$
- $X = \{0^\omega\}$.



Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

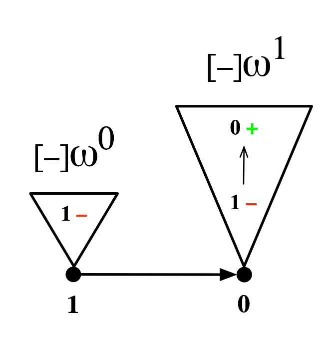
Example (continued)

The linked pairs are: $(0,0)$, $(0,1)$, and $(1,1)$.

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by

$$\begin{array}{ll} 0 \cdot 0 = 0 & 0 \cdot 1 = 0 \\ 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega \\ 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$

- $X = \{0^\omega\}$.



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Example (continued)

The linked pairs are: $(0, 0)$, $(0, 1)$, and $(1, 1)$.

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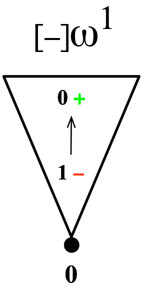
- $X = \{0^\omega\}$.



Therefore $d_{SG}((S, X)) = d_W(K) = \omega$, and K is non-self-dual.

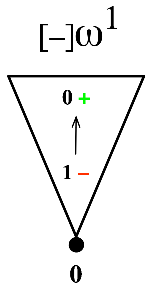
Example (continued)

The linked pairs are: $(0,0)$, $(0,1)$, and $(1,1)$.

- $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ defined by

$$\begin{array}{ll} 0 \cdot 0 = 0 & 0 \cdot 1 = 0 \\ 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega \\ 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$

- $X = \{0^\omega\}$.



Therefore $d_{SG}((S,X)) = d_W(K) = \omega$, and K is non-self-dual.

Example

Consider the syntactic pointed ω-semigroup of $L = (a\{b,c\}^* \cup \{b\})^\omega$:

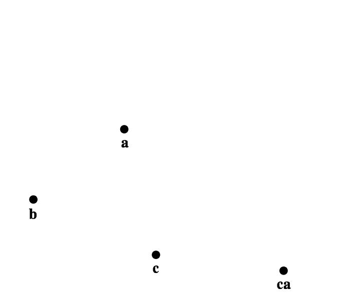
- $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is defined by

$$
\begin{array}{cccc}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega
\end{array}
$$

- $Y = \{a^\omega\}$.

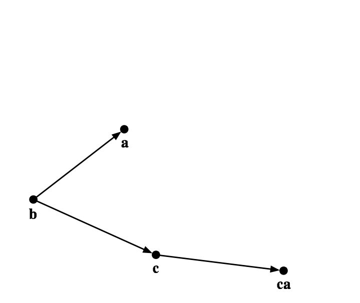
Example (continued)

The linked pairs are: (b,b), (a,b), (a,a), (a,c), (a,ca), (c,b), (c,c), (ca,b), (ca,a), (ca,c), (ca,ca).



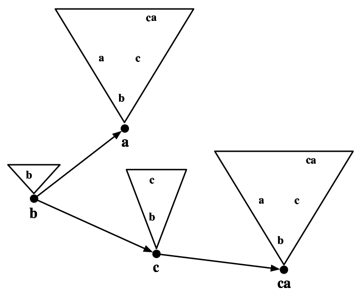
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



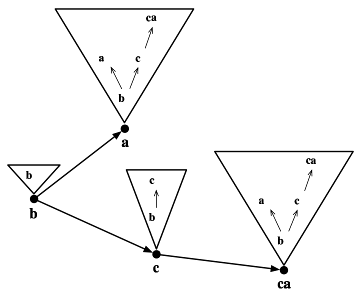
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



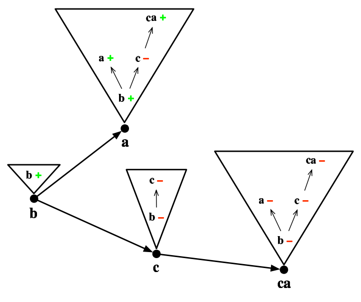
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



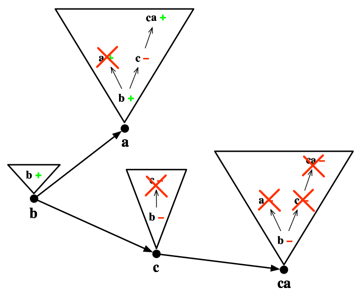
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



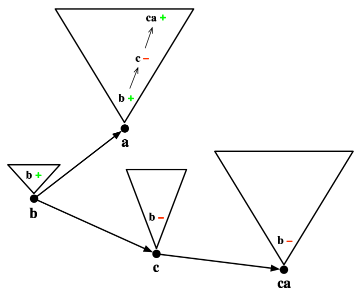
Example (continued)

The linked pairs are: (b,b), (a,b), (a,a), (a,c), (a,ca), (c,b), (c,c), (ca,b), (ca,a), (ca,c), (ca,ca).



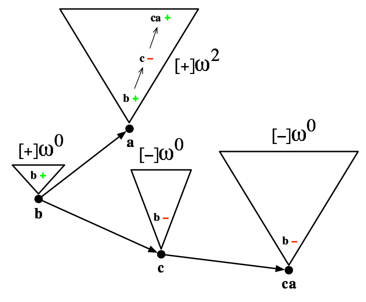
Example (continued)

The linked pairs are: (b,b), (a,b), (a,a), (a,c), (a,ca), (c,b), (c,c), (ca,b), (ca,a), (ca,c), (ca,ca).



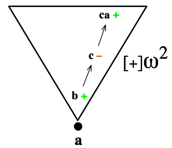
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



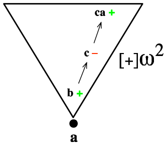
Example (continued)

The linked pairs are: (b, b), (a, b), (a, a), (a, c), (a, ca), (c, b), (c, c), (ca, b), (ca, a), (ca, c), (ca, ca).



Example (continued)

The linked pairs are: (b,b), (a,b), (a,a), (a,c), (a,ca), (c,b), (c,c), (ca,b), (ca,a), (ca,c), (ca,ca).

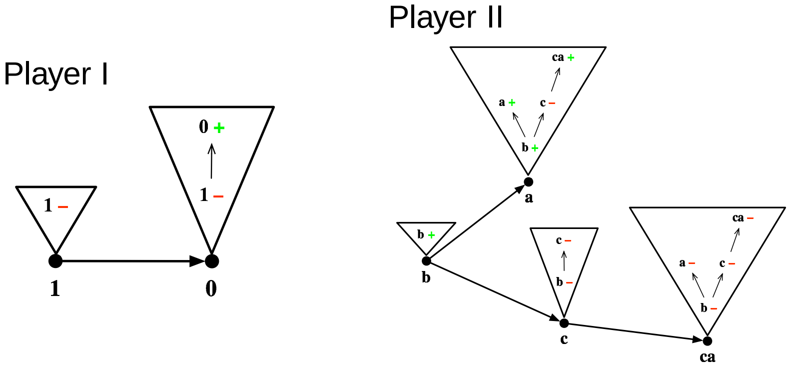


Therefore $d_{SG}((T,Y)) = d_W(L) = \omega^2$, and L is non-self-dual.

Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X) I

(T,Y) II



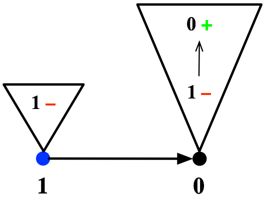
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

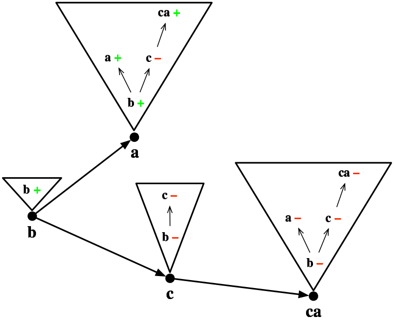
(S,X) I 1

(T,Y) II

Player II

Player I

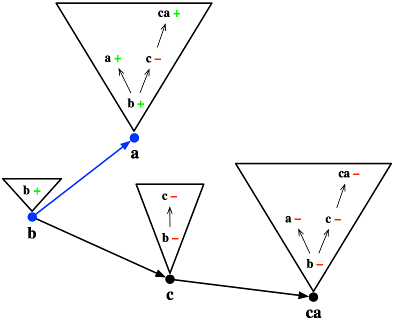




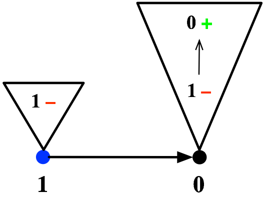
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**
(T,Y)	**II**	***a***

Player II



Player I

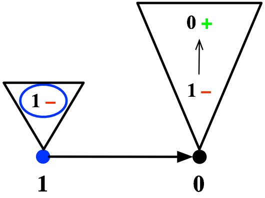


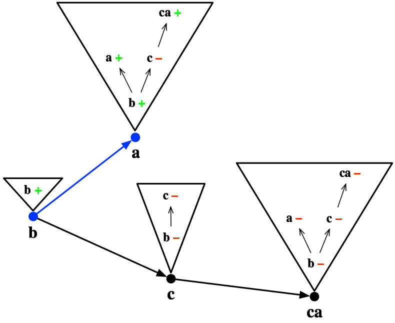
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	

Player II

Player I



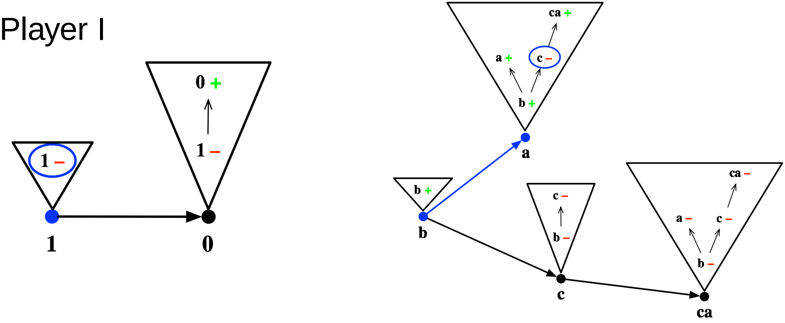


Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

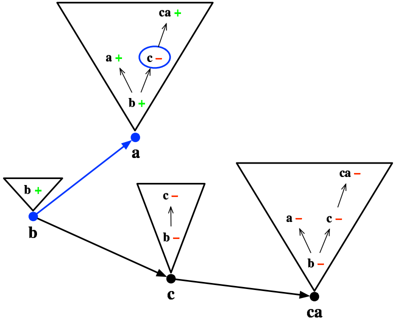
Player I



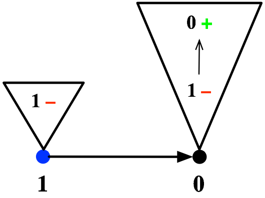
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II



Player I

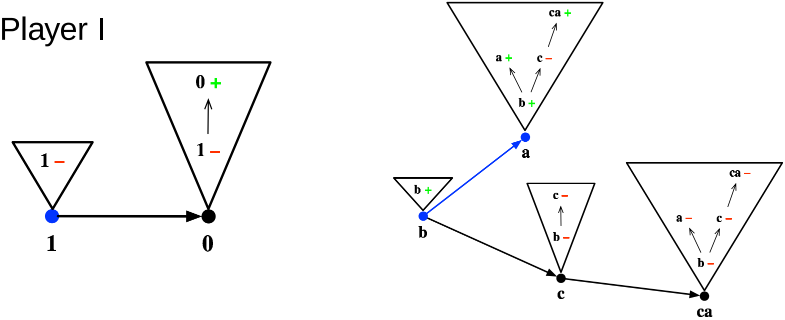


Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**
(T,Y)	**II**	***a***	***c***

Player II

Player I

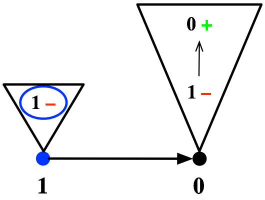


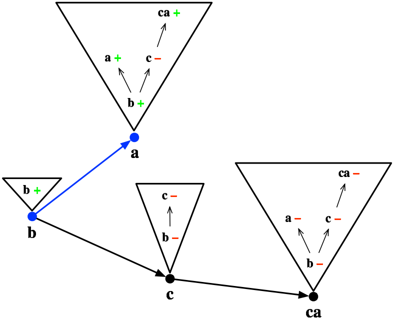
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	

Player II

Player I

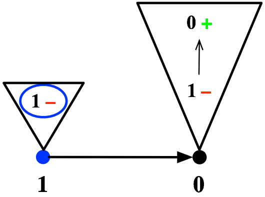




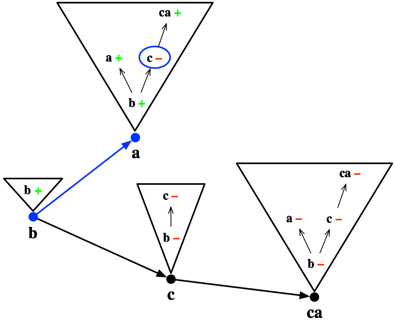
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player I



Player II

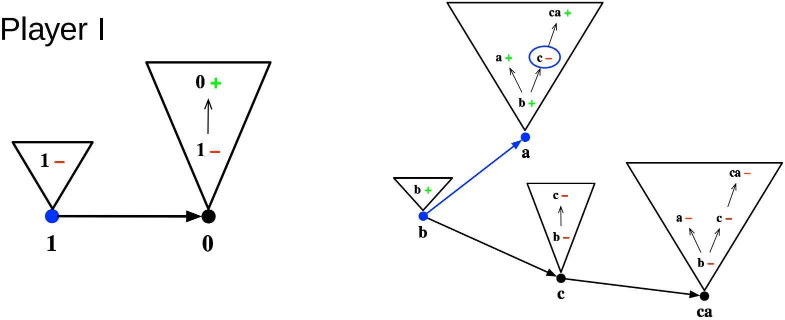


Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player II

Player I

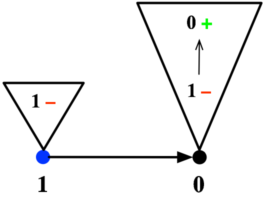


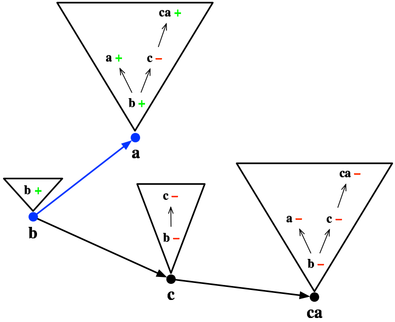
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**
(T,Y)	**II**	***a***	***c***	***c***

Player II

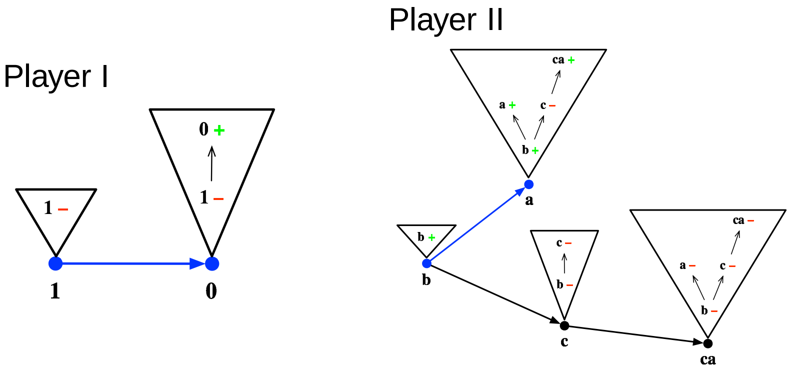
Player I





Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

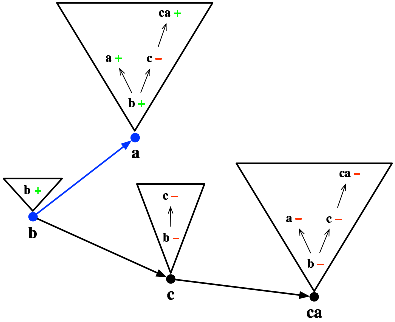
(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	



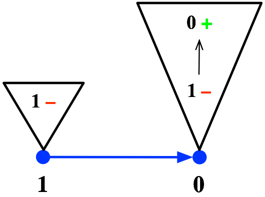
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**

Player II



Player I

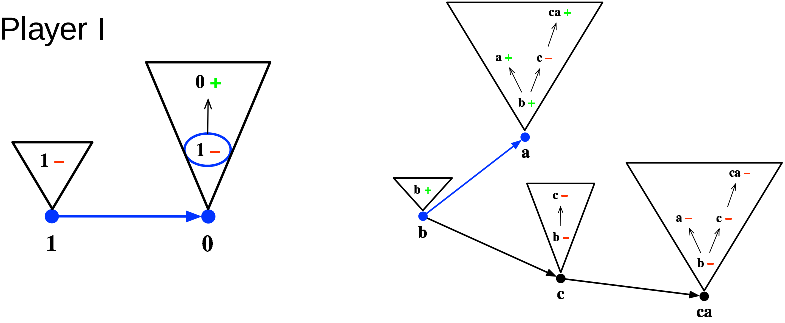


Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	

Player II

Player I

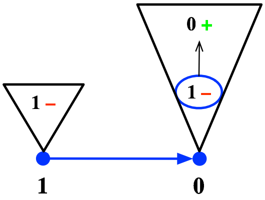


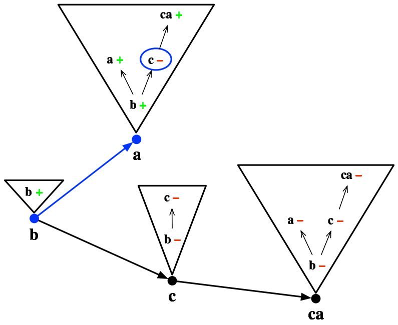
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

Player II

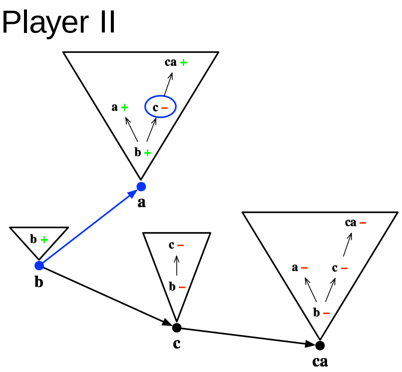
Player I

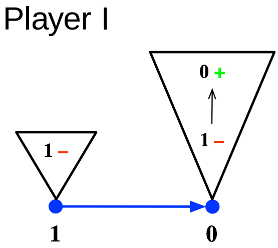




Therefore $(S,X) \leq_{SG} (T,Y)$, or equivalently $K \leq_W L$.

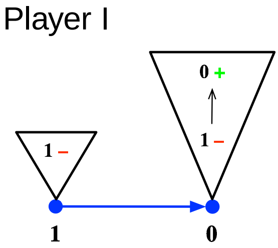
(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***

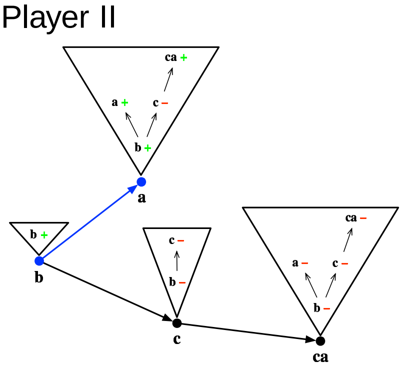




Therefore $(S,X) \leq_{SG} (T,Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***



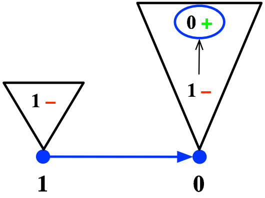


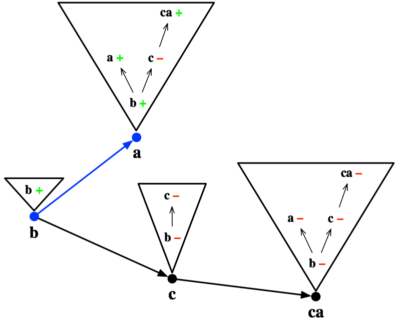
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	

Player II

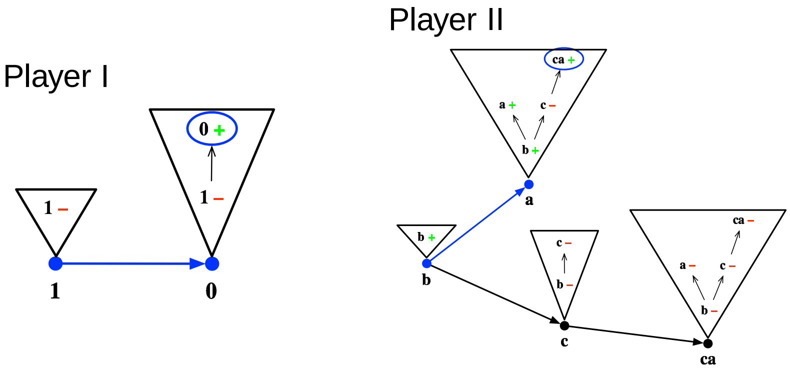
Player I





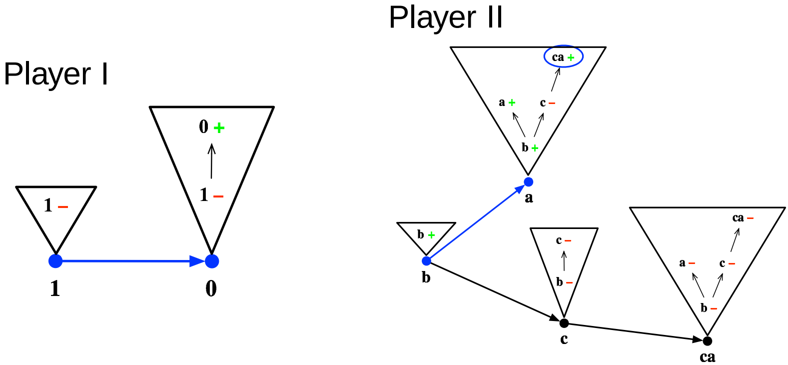
Therefore $(S,X) \leq_{SG} (T,Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	1	1	1	0	1	0
(T,Y)	**II**	*a*	*c*	*c*	–	*c*	*ca*



Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

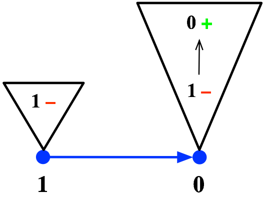
(S,X)	**I**	1	1	1	0	1	0
(T,Y)	**II**	*a*	*c*	*c*	–	*c*	*ca*



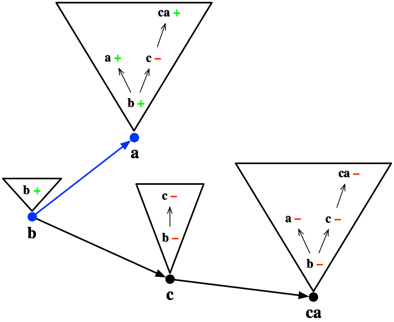
Therefore $(S,X) \leq_{SG} (T,Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***

Player I



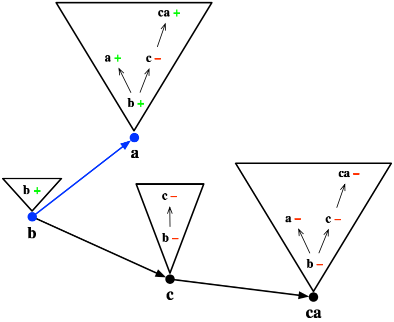
Player II



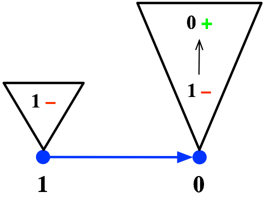
Therefore $(S, X) \leq_{SG} (T, Y)$, or equivalently $K \leq_W L$.

(S,X)	**I**	**1**	**1**	**1**	**0**	**1**	**0**	**……**
(T,Y)	**II**	***a***	***c***	***c***	**–**	***c***	***ca***	**……**

Player II



Player I



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- Decidability procedure of the $\mathbb{FSG}$-hierarchy.
- **One can compute the Wagner degree of an ω-rational language directly on its syntactic image.**
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- Computer science: towards a similar analysis of the algebraic counterpart of more complex abstract machines.
- Descriptive set theory: towards an algebraic study of the Wadge hierarchy.
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