An Infinite Game on ω-Semigroups

joint work with Jacques Duparc

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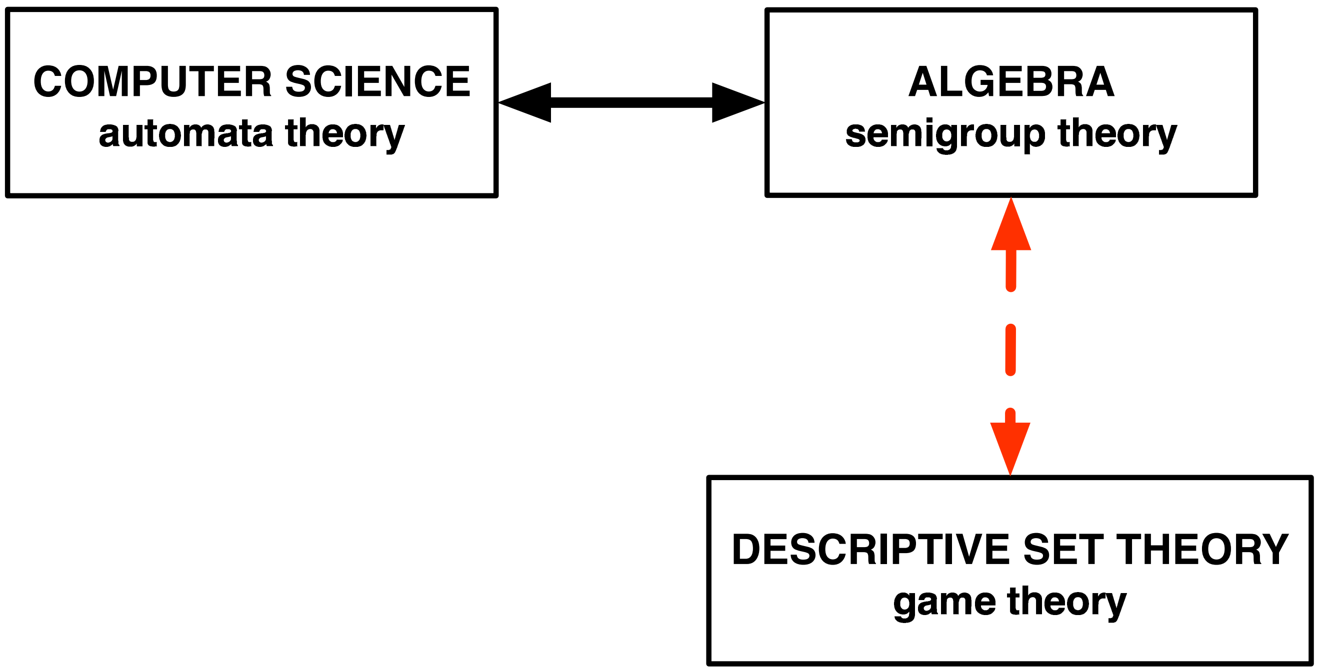
28/11/2004


COMPUTER SCIENCE
automata theory
ALGEBRA
semigroup theory
DESCRIPTIVE SET THEORY
game theory

COMP. SCIENCE		**ALGEBRA**
automaton **(rational language)**	$\longleftrightarrow$	**finite semigroup**
Büchi automaton **(ω-rational language)**	$\longleftrightarrow$	**Wilke algebra** **finite $\boldsymbol{\omega}$-semigroup**

$\boldsymbol{\omega}$***-semigroup*** $\boldsymbol{S = (S_+, S_\omega)}$ **(J.-É. Pin)**

- $(\boldsymbol{S_+}, \cdot)$ **is a semigroup,** $\boldsymbol{S_\omega}$ **is a set**

- $\boldsymbol{\pi : {S_+}^\omega \longrightarrow S_\omega}$ **an infinite product** ω**-associative**

  $\pi(s_0, s_1, s_2, \ldots) = \pi(s_0 \cdot s_1 \cdot \ldots \cdot s_{n_0}, s_{n_0+1} \cdot \ldots \cdot s_{n_1}, \ldots)$

part. case : ***free*** ω***-semigroup*** $S = (S_+, S_\omega = S_+^\omega)$

where the infinite prod. is the identity

a reduction relation $\leq_{SG}$ **on** ω**-semigroups**

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$

$$\boxed{\begin{array}{lll} \boldsymbol{X \leq_{SG} Y} & \Leftrightarrow_{def} & X \textbf{ is "less complicated" than } Y \\ & & \textbf{i.e. } \exists \textbf{ "simple" } f \textbf{ s.t. } (u \in X \Leftrightarrow f(u) \in Y) \\ & \Leftrightarrow_{def} & \textbf{II has a w.s. in the game } SG(X,Y) \end{array}}$$

so "$\leq_{SG}$**" is a bridge between alg. and game theory!**

An infinite two players game $\boldsymbol{SG(X,Y)}$ *on* ω*-semigroups*

Let $S = (S_+, S_\omega)$ **,** $T = (T_+, T_\omega)$ **be two** ω**-sg and** $X \subseteq S_\omega$ **,** $Y \subseteq T_\omega$**.**

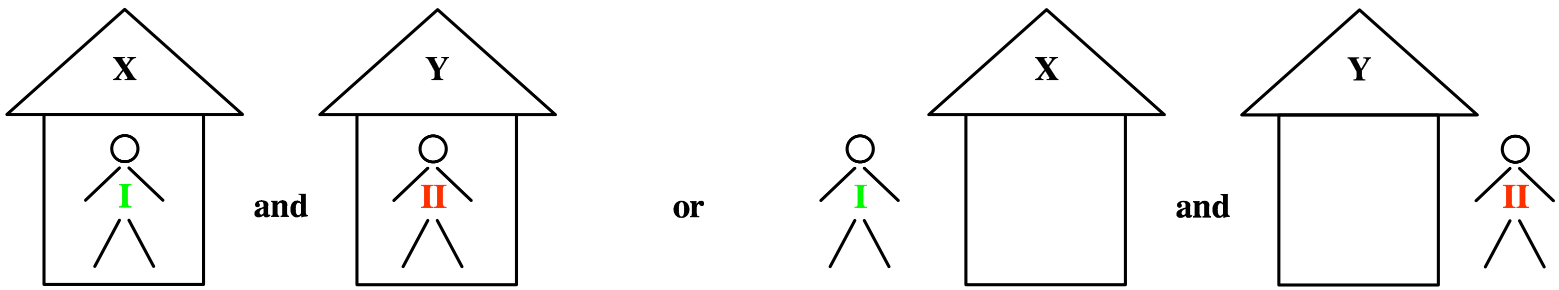
(X)	**I**	s_0	s_1	$\ldots$	$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle s_0, s_1, s_2, \ldots \rangle$
(Y)	**II**	t_0	t_1	$\ldots$	$\xrightarrow{\text{after } \omega \text{ moves}}$	$\langle t_0, t_1, t_2, \ldots \rangle$

II wins

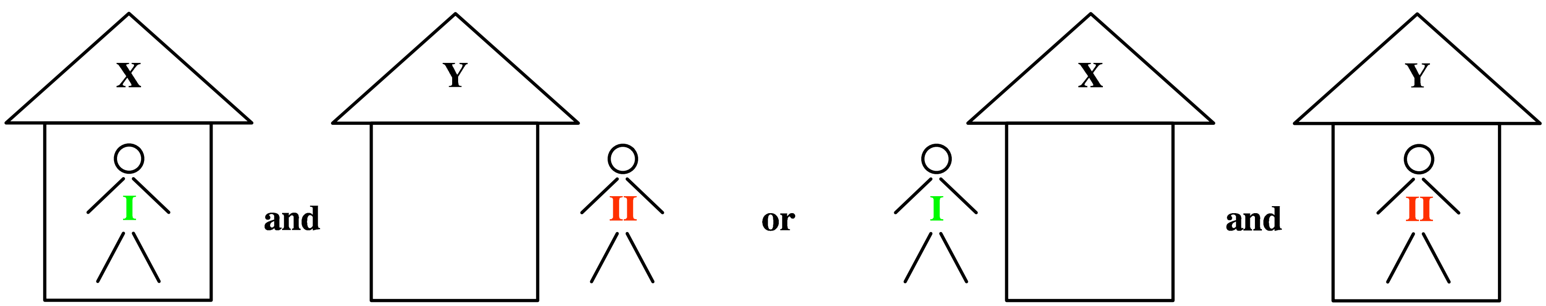
$$\Leftrightarrow_{def}$$

$$\boldsymbol{\big(\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y\big)}$$

II wins in $SG(X, Y)$ iff



I wins in $SG(X, Y)$ iff



use Borel determinacy:

- ***a priori***, "$\leq_{SG}$" is:

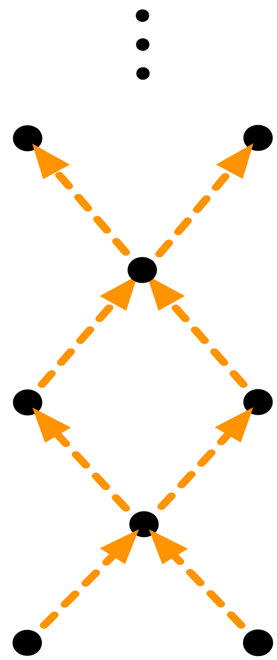
  - only a partial ordering ...

- ***by Borel determinacy***, "$\leq_{SG}$" is:

  - partial ordering

  - well founded

  - antichains of length at most 2



topology on S_ω:

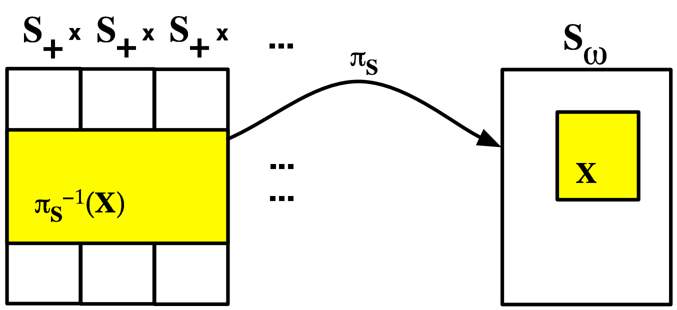
- a ***bad*** **idea:**

$$X \subseteq S_\omega \textbf{ basic open } \Leftrightarrow_{def} X = s \cdot S_\omega \,,\; s \in S_+$$

when S_+ **is a group, only two Borel sets** $= \{\emptyset, S_\omega\}$

- a ***better*** **idea:**

$$\color{blue}{X} \subseteq S_\omega \ \color{blue}{\textbf{basic open}} \Leftrightarrow_{def} \pi_S^{-1}(X) \textbf{ open of } S_+^\omega$$



S_+^ω **equipped with the product top of the discrete top on** S_+

Properties of the SG-hierarchy:

- **when restricted to free ω-semigroups:**

  SG-hierarchy $\equiv$ Wadge hierarchy

- **when restricted to finite ω-semigroups:**

  SG-hierarchy $\equiv$ Wagner hierarchy

Proposition:

let $S = (S_+, S_\omega)$ **be an** ω**-semigroup and** $X \subseteq S_\omega$

$$\boxed{\begin{array}{c} X \not\leq_{SG} X^C \\ \Leftrightarrow \\ S_+ \text{ "is" a monoid} \\ \Leftrightarrow \\ \text{player in charge of } X \text{ is allowed to skip} \end{array}}$$

Proposition:

let $S = (S_+, S_\omega)$ **be an** ω**-semigroup and** $X \subseteq S_\omega$

$$\boxed{\begin{array}{c} X \equiv_{SG} s \cdot X, \ \forall s \in S_+ \\ \Leftrightarrow \\ S_+ \text{ \textbf{"is" a group}} \\ \Leftrightarrow \\ \text{\textbf{player in charge of} } X \text{ \textbf{is allowed to erase}} \end{array}}$$

- **important hierarchies are particular cases of SG-hierarchy**
- **essential algebraic notions expressed in a very natural game theoretical way**
- **work in progress ...**