Expressive power of Nondeterministic Evolving Recurrent Neural Networks in Terms of their Attractor Dynamics

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Introduction

- We assume that some aspect of information processing in the brain can be approached from the perspective of computability theory.
- The computational capabilities of recurrent neural networks have mainly been studied in the context of classical computation (McCulloch & Pitts, Turing, Kleene, von Neumann, Minsky, Papert,..., Siegelmann & Sontag,...).
- Here, we introduce a natural model of nondeterministic recurrent neural networks and study their expressive power in terms of their attractor dynamics, i.e., in the context of infinite (or non-terminating) computations.

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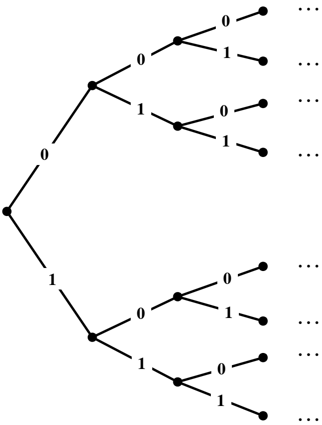
Topology

The Cantor space $\{0,1\}^\omega$
the set of infinite sequences of bits



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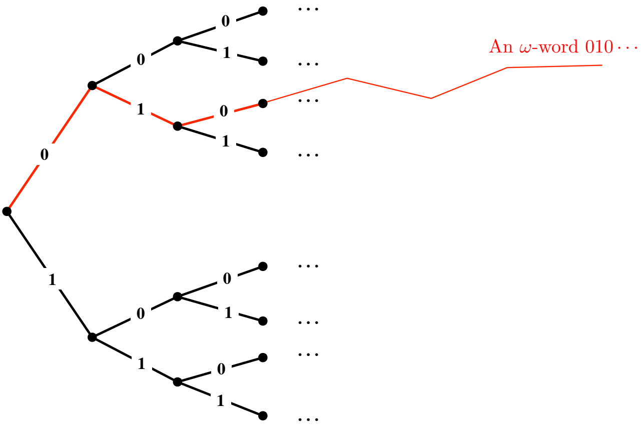

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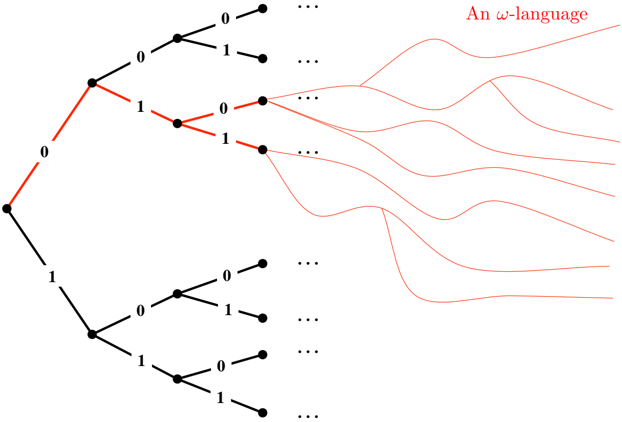

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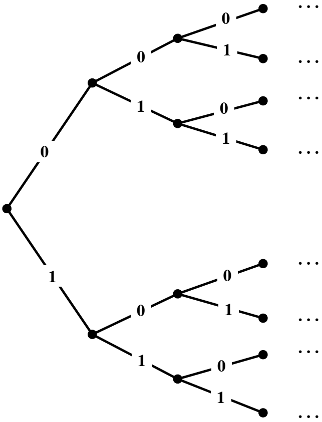
equipped with product topology
of the discrete topology on $\{0,1\}$



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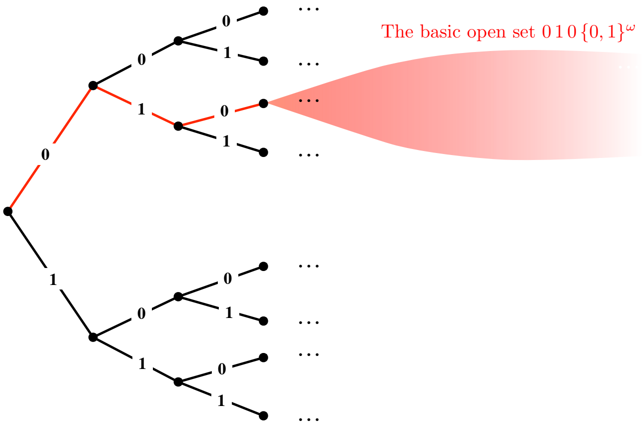

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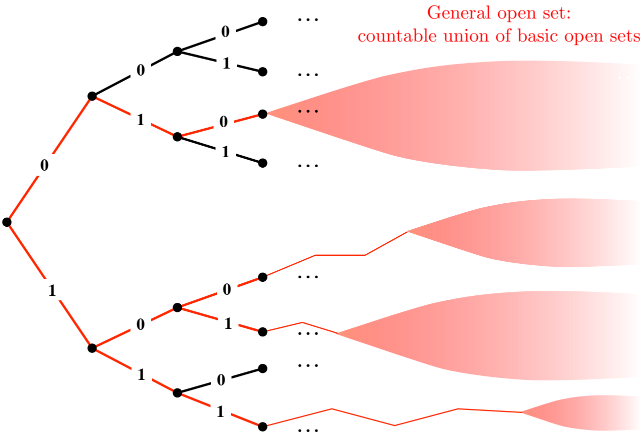
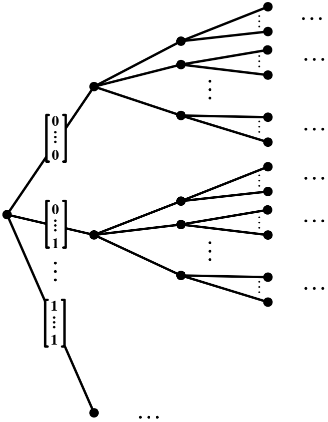


Topology

The space $(\mathbb{B}^N)^\omega$
the set of infinite sequences of N-dim. Boolean vectors



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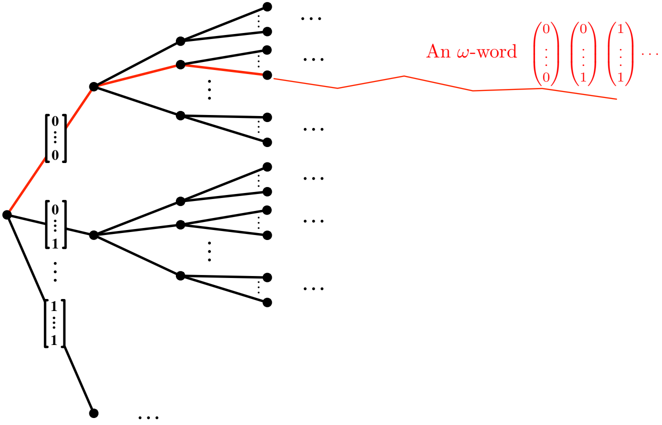
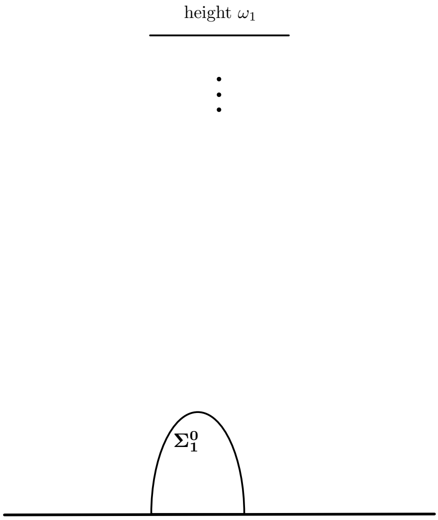


Topology

height ω_1

$\vdots$

Topology



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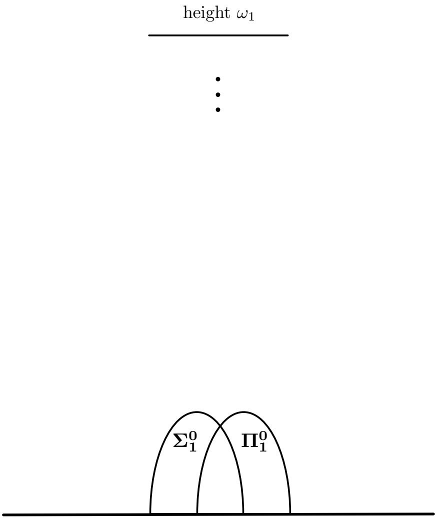

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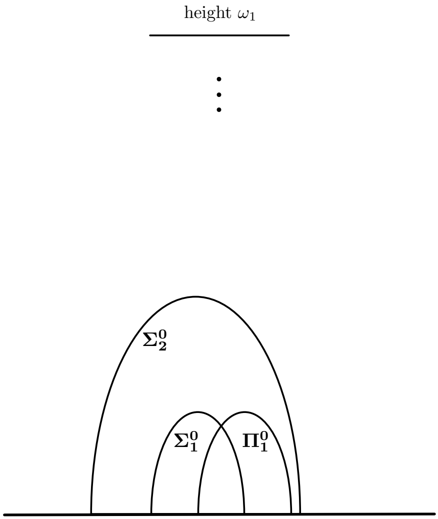
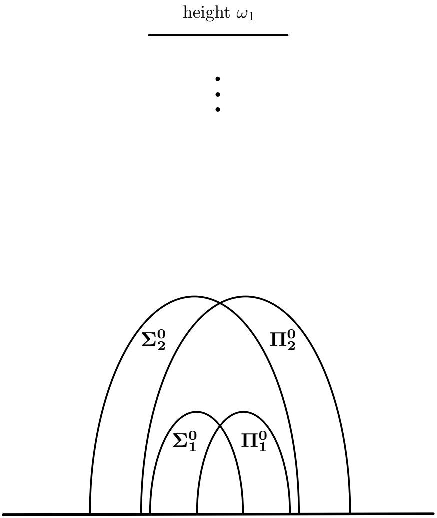

height ω_1
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$

Topology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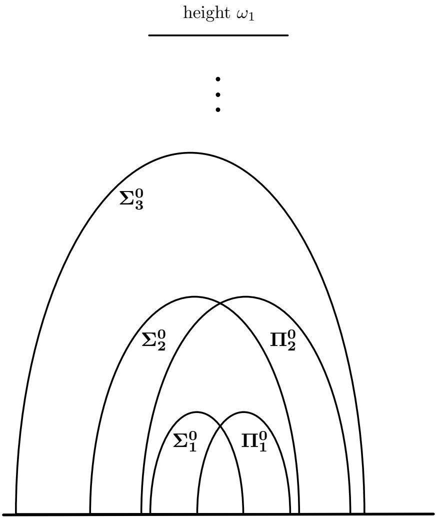
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$\boldsymbol{\Sigma}^0_1$
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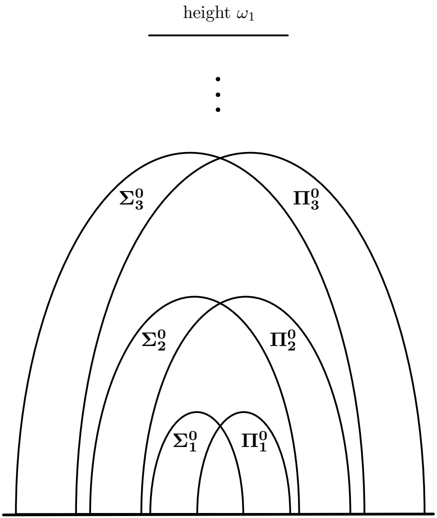

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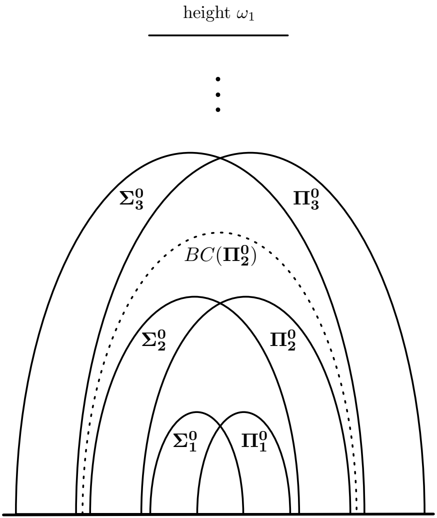

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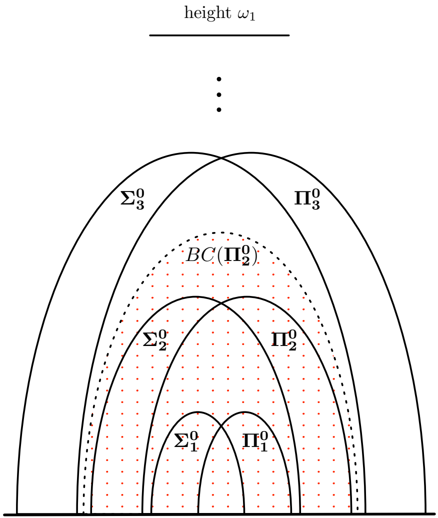
$\vdots$
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$BC(\mathbf{\Pi}^0_2)$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$

Topology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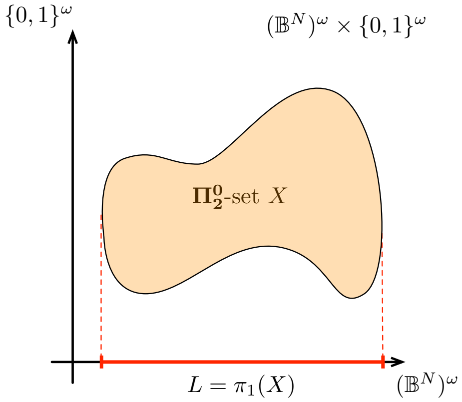
$\mathbf{\Pi}^0_1$

Topology

$$\begin{tikzcd}[column sep=small]
 & \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
 & \boldsymbol{\Delta}^0_3 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\supsetneq"] & \\
\boldsymbol{\Sigma}^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_2 \arrow[ul, phantom, sloped, "\supsetneq"] \\
 & \boldsymbol{\Delta}^0_2 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\supsetneq"] & \\
\boldsymbol{\Sigma}^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_1 \arrow[ul, phantom, sloped, "\supsetneq"] \\
 & \boldsymbol{\Delta}^0_1 \arrow[ul, phantom, sloped, "\subsetneq"] \arrow[ur, phantom, sloped, "\supsetneq"] &
\end{tikzcd}$$

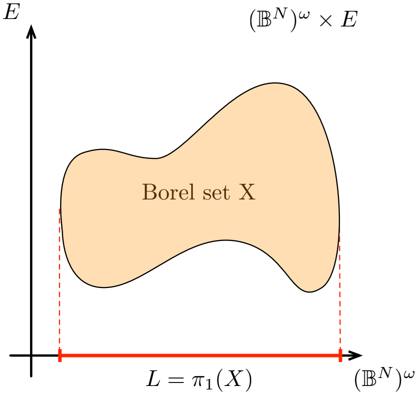
Topology

- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\boldsymbol{\Sigma}_1^1$) iff it is the first projection of some $\boldsymbol{\Pi}_2^0$-set $X \subseteq (\mathbb{B}^N)^\omega \times \{0,1\}^\omega$.



Topology

- An ω-language $L \subseteq (\mathbb{B}^N)^\omega$ is *analytic* ($\mathbf{\Sigma}^1_1$) iff it is the first projection of some Borel set $X \subseteq (\mathbb{B}^N)^\omega \times E$, where E is a Polish space.



Topology

$$
\begin{tikzcd}[column sep=small]
 & \vdots & \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
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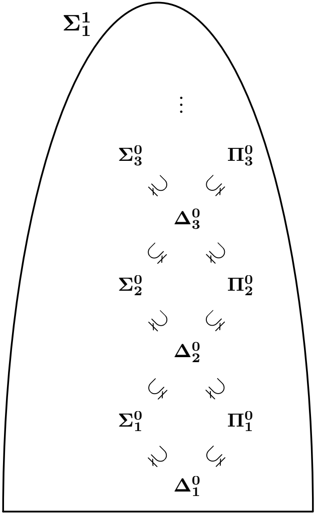
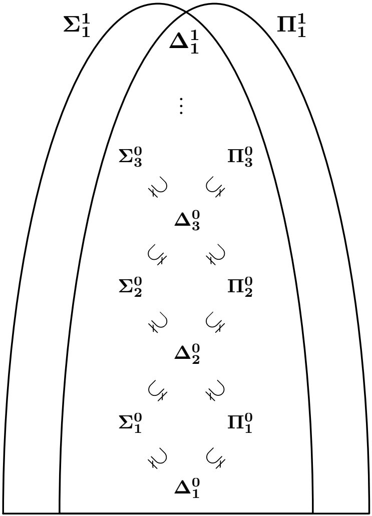
\boldsymbol{\Sigma}^0_1 \arrow[ur, hook] & & \boldsymbol{\Pi}^0_1 \arrow[ul, hook] \\
 & \boldsymbol{\Delta}^0_1 \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Topology


$\mathbf{\Sigma}^1_1$
$\vdots$
$\mathbf{\Sigma}^0_3$ $\mathbf{\Pi}^0_3$
$\mathbf{\Delta}^0_3$
$\mathbf{\Sigma}^0_2$ $\mathbf{\Pi}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Sigma}^0_1$ $\mathbf{\Pi}^0_1$
$\mathbf{\Delta}^0_1$

Topology


$\mathbf{\Sigma}_1^1$
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$\vdots$
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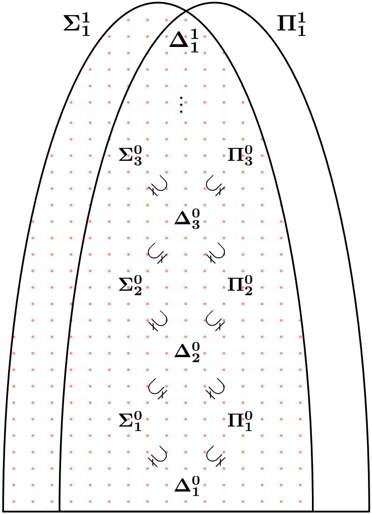
Topology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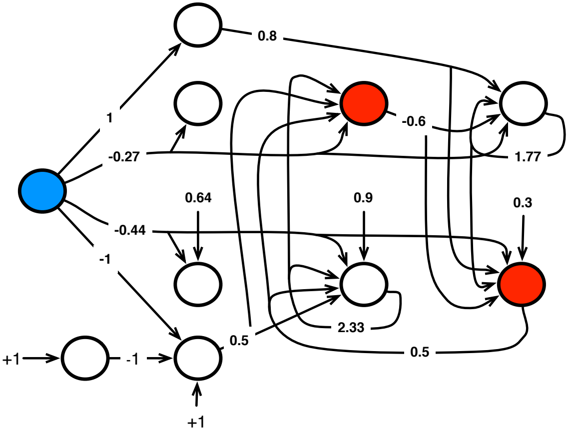
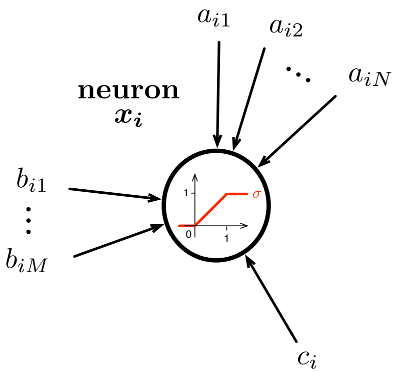
$\mathbf{\Delta}^0_1$

Recurrent neural network

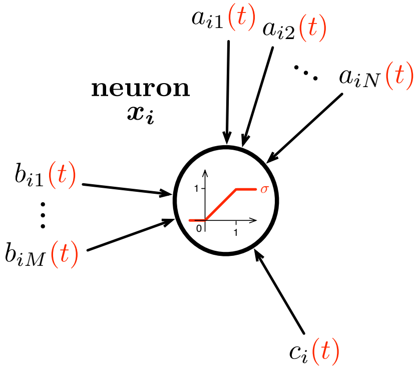


Static recurrent neural networks



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

Evolving recurrent neural networks



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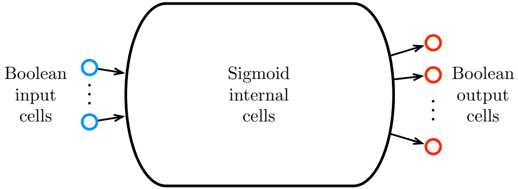
Hybrid recurrent neural networks

We consider RNNs with Boolean input cells, sigmoid internal cells, Boolean output cells, and working on infinite input streams.



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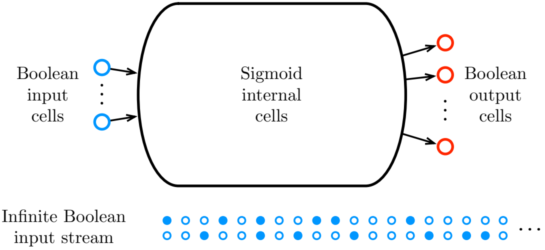

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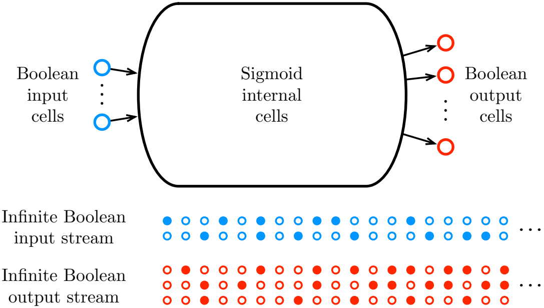

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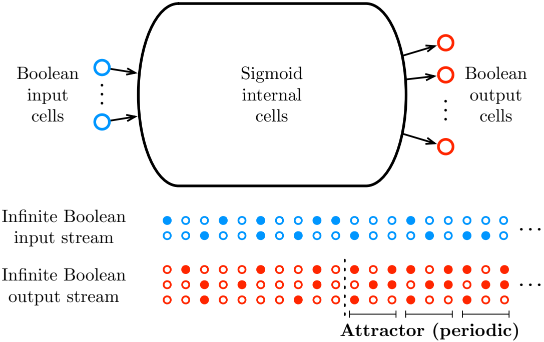

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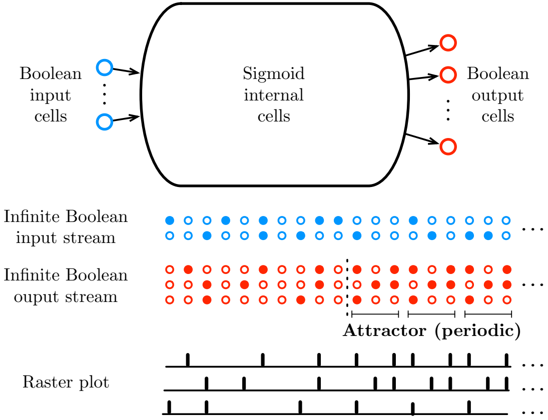

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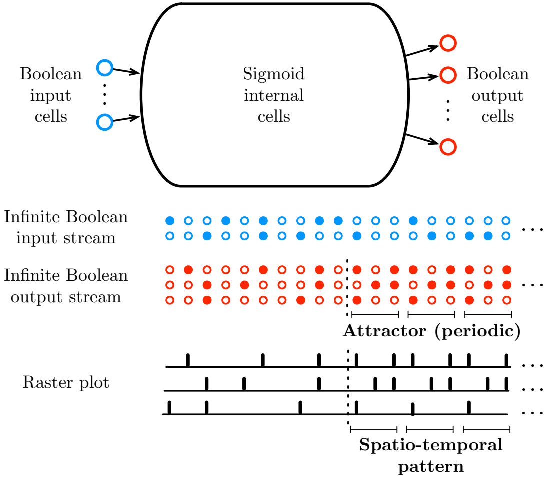

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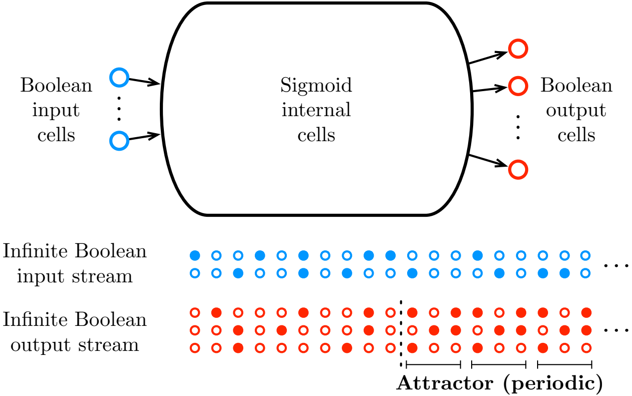
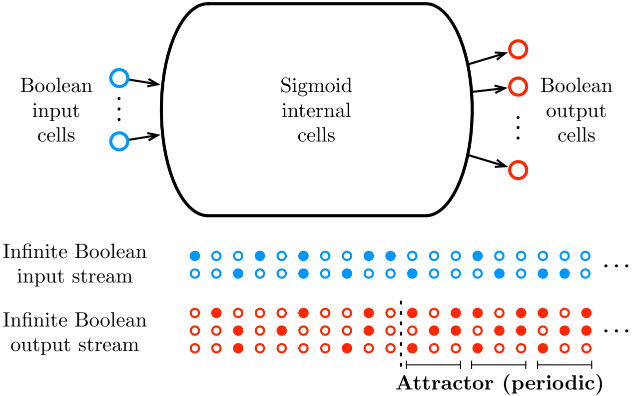


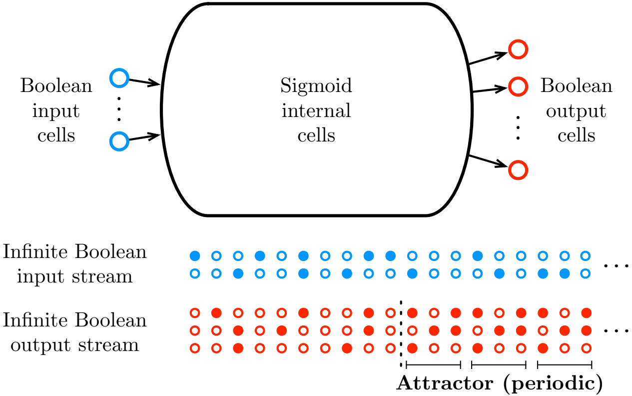
- The attractors are assumed to be classified into two possible kinds: *meaningful* or *spurious*.

Hybrid recurrent neural networks



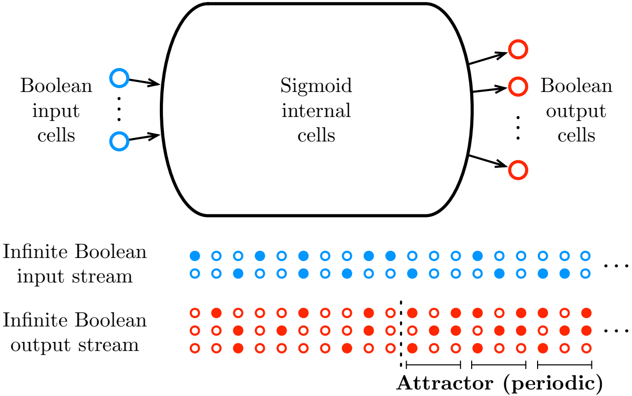
- An infinite Boolean input stream is *accepted* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *meaningful* attractor.

Hybrid recurrent neural networks



- An infinite Boolean input stream is *rejected* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *spurious* attractor.

Hybrid recurrent neural networks

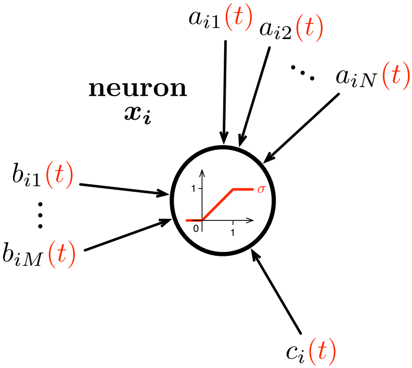


- The set of all input streams that are accepted by $\mathcal{N}$ is the *ω-language* recognized by $\mathcal{N}$.

Results

	Static	Bi-valued Evolving	Evolving
$\mathbb{Q}$	Turing (Muller) $\in BC(\mathbf{\Pi^0_2})$	super-Turing $= BC(\mathbf{\Pi^0_2})$	super-Turing $= BC(\mathbf{\Pi^0_2})$
$\mathbb{R}$	super-Turing $= BC(\mathbf{\Pi^0_2})$	super-Turing $= BC(\mathbf{\Pi^0_2})$	super-Turing $= BC(\mathbf{\Pi^0_2})$

Evolving recurrent neural networks



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Nondeterminism

Suppose that the neural net contains K evolving synaptic weights:

- A K-dimensional vector $\vec{w}(t) = (w_1(t), \dots, w_K(t))^T \in \mathbb{Q}^K$ or $\mathbb{R}^K$ describes the value of the K synaptic weights at time t.
- A *possible evolution* is an infinite sequence
$$e = \vec{w}(0)\,\vec{w}(1)\,\vec{w}(2) \cdots \in (\mathbb{Q}^K)^\omega \text{ or } (\mathbb{R}^K)^\omega$$
which describes the synaptic weights at successive time steps.
- We suppose that each network is provided with a corresponding *evolution set*: a closed set $E \subseteq (\mathbb{Q}^K)^\omega$ or $(\mathbb{R}^K)^\omega$ describing all possible evolutions of the net.
- **Nondeterminism:** *At the beginning of the computation, the network selects one possible evolution $e \in E$, and sticks to it throughout its whole computational process.*

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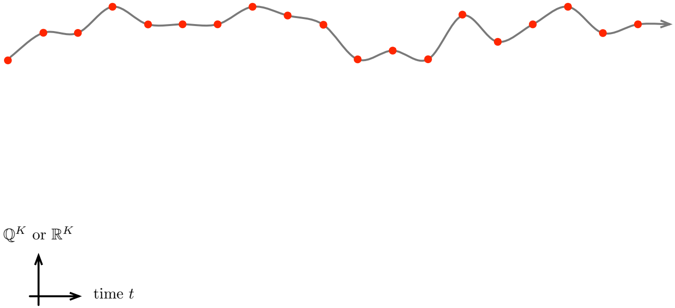

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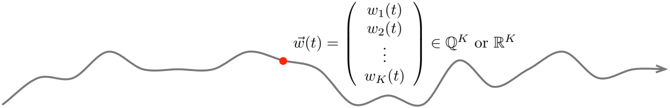
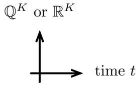
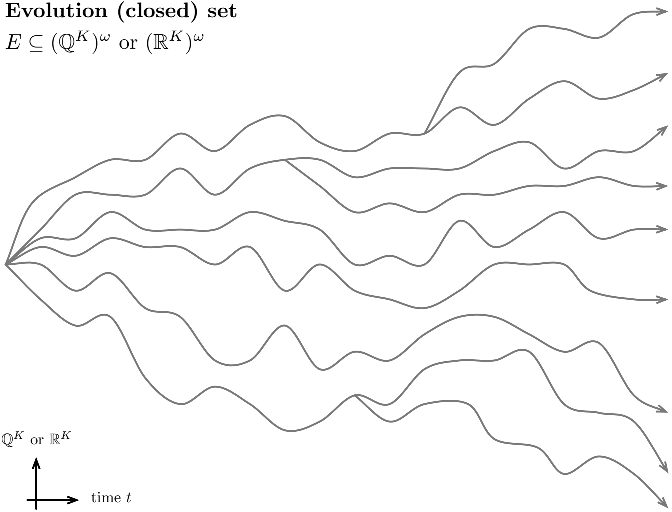




Nondeterminism

Evolution (closed) set

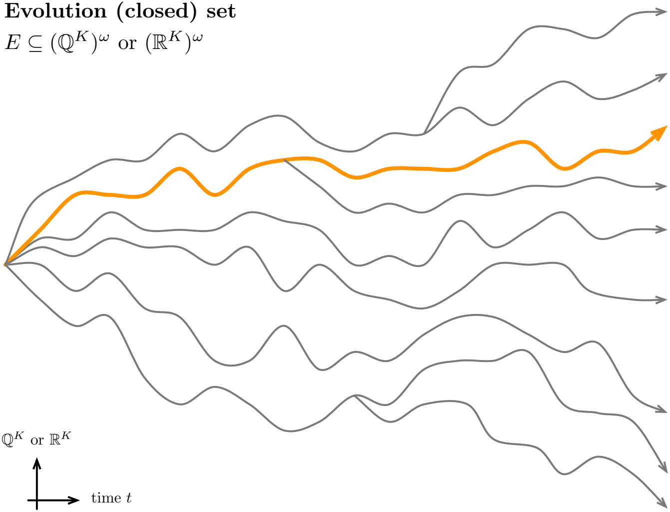
$E \subseteq (\mathbb{Q}^K)^\omega$ or $(\mathbb{R}^K)^\omega$



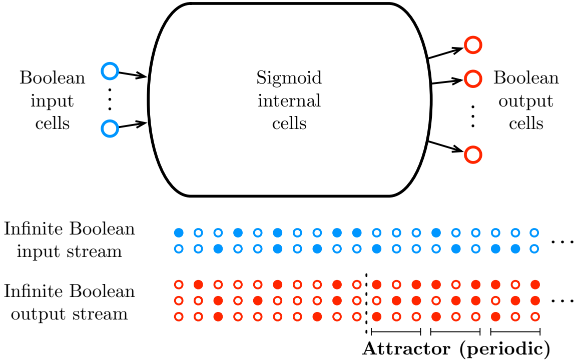
Nondeterminism

Evolution (closed) set

$E \subseteq (\mathbb{Q}^K)^\omega$ or $(\mathbb{R}^K)^\omega$

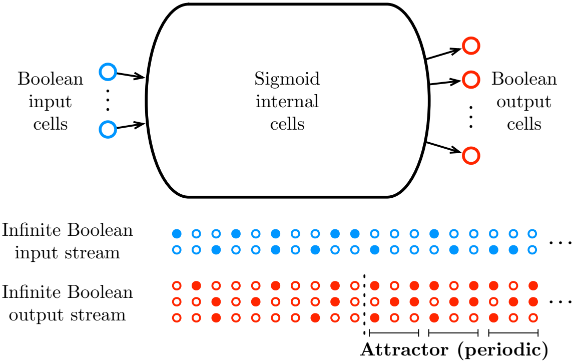


Nondet. hybrid recurrent neural networks



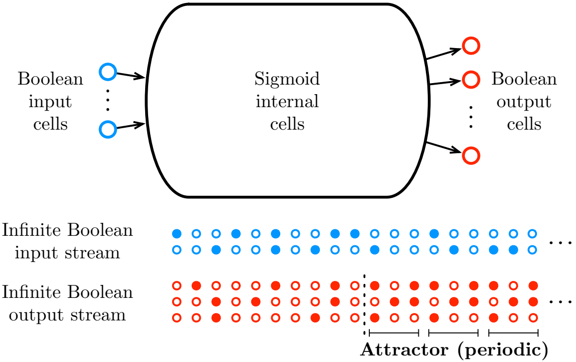
- The attractors are assumed to be classified into two possible kinds: *meaningful* or *spurious*.

Nondet. hybrid recurrent neural networks



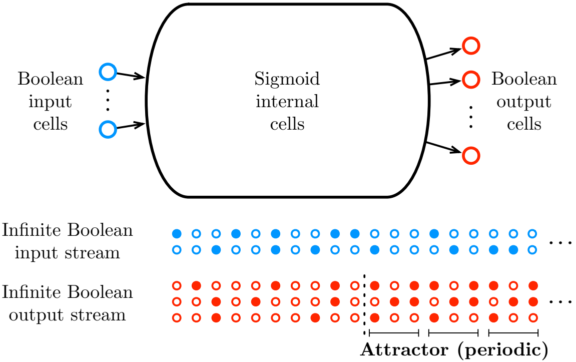
- An infinite Boolean input stream is *accepted* by $\mathcal{N}$ if *there exists a possible evolution* $e \in E$ such that the corresponding Boolean output stream visits a *meaningful* attractor.

Nondet. hybrid recurrent neural networks



- An infinite Boolean input stream is *rejected* by $\mathcal{N}$ if *for all possible evolution* $e \in E$, the corresponding Boolean output stream visits a *spurious* attractor.

Nondet. hybrid recurrent neural networks



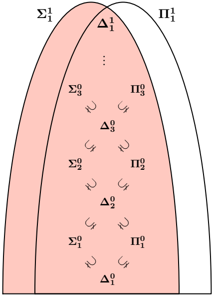
- The set of all input streams that are accepted by $\mathcal{N}$ is the ω-*language* recognized by $\mathcal{N}$.

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \mathbf{\Sigma}^1_1$
- *L is recognizable by some nondeterministic evolving RNN[$\mathbb{Q}$]*
- *L is recognizable by some nondeterministic evolving RNN[$\mathbb{R}$]*



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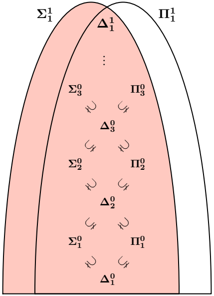

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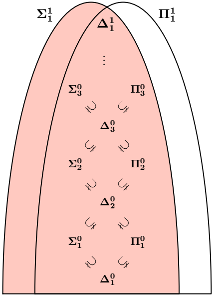

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\mathbf{\Sigma}^1_1 \\
\rotatebox{90}{$\subseteq$} \\
\mathcal{L}(N\!-\!Ev\!-\!RNN[\mathbb{R}]s) \\
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Results

Proposition

Let $L \in \mathbf{\Sigma}^1_1$. Then L is recognizable by some nondet. evolving RNN[$\mathbb{Q}$].

Proof (sketch):

- Since $L \in \Sigma^1_1$, there exists some Π^0_2 set $X \subseteq (\mathbb{B}^M)^\omega \times \{0,1\}^\omega$ such that $L = \pi_1(X)$.
- X can thus be encoded into some infinite word $w_X \in \{0,1\}^\omega$.
- We can then build an nondet. evolving RNN[$\mathbb{Q}$] – *whose one of its evolving synaptic weights follows w_X* – which, on input $s \in (\mathbb{B}^M)^\omega$ and evolution $e \in \{0,1\}^\omega$, enters a meaningful attractor iff $(s, e) \in X$.
- It follows that $L(\mathcal{N}) = \pi_1(X) = L$.

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- It follows that $L(\mathcal{N}) = \pi_1(X) = L$.

$\square$

Results

Proposition

Let $\mathcal{N}$ be some nondet. evolving RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma^1_1}$.

Proof (sketch):

- The function $f_{\mathcal{N}} : (\mathbb{B}^M)^\omega \times E \to (\mathbb{B}^P)^\omega$ associated with the dynamics of $\mathcal{N}$ is of Baire class 1 (preimage of a Σ^0_1 is a Σ^0_2).
- Accordingly, the ω-language $L(\mathcal{N})$ can be expressed as the first projection of a finite Boolean combination of $\mathbf{\Sigma^0_3}$ and $\mathbf{\Pi^0_3}$ sets (i.e., of a Borel set) of the Polish space $(\mathbb{B}^M)^\omega \times E$.
- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma^1_1}$.

Results

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Let $\mathcal{N}$ be some nondet. evolving RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma}^1_1$.

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- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma}^1_1$. $\square$

Results

Proposition

Let $\mathcal{N}$ be some nondet. evolving RNN[$\mathbb{R}$]. Then $L(\mathcal{N}) \in \mathbf{\Sigma}_\mathbf{1}^\mathbf{1}$.

Proof (sketch):

- The function $f_\mathcal{N} : (\mathbb{B}^M)^\omega \times E \to (\mathbb{B}^P)^\omega$ associated with the dynamics of $\mathcal{N}$ is of Baire class 1 (preimage of a $\mathbf{\Sigma}_1^0$ is a $\mathbf{\Sigma}_2^0$).
- Accordingly, the ω-language $L(\mathcal{N})$ can be expressed as the first projection of a finite Boolean combination of $\mathbf{\Sigma_3^0}$ and $\mathbf{\Pi_3^0}$ sets (i.e., of a Borel set) of the Polish space $(\mathbb{B}^M)^\omega \times E$.
- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma}_1^1$. $\square$

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- Therefore, $L(\mathcal{N}) \in \mathbf{\Sigma_1^1}$. $\square$

Results

	Bi-valued Evolving	Evolving
$\mathbb{Q}$	super-Turing $= \mathbf{\Sigma^1_1}$	super-Turing $= \mathbf{\Sigma^1_1}$
$\mathbb{R}$	super-Turing $= \mathbf{\Sigma^1_1}$	super-Turing $= \mathbf{\Sigma^1_1}$

Summary

DET.	Static	Bi-valued Evolving	Evolving
$\mathbb{Q}$	Turing (Muller) $\in BC(\mathbf{\Pi_2^0})$	super-Turing $= BC(\mathbf{\Pi_2^0})$	super-Turing $= BC(\mathbf{\Pi_2^0})$
$\mathbb{R}$	super-Turing $= BC(\mathbf{\Pi_2^0})$	super-Turing $= BC(\mathbf{\Pi_2^0})$	super-Turing $= BC(\mathbf{\Pi_2^0})$

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Conclusion

- We provided a characterization of the expressive power of recurrent neural networks in terms of their attractor dynamics.
- In general, the super-Turing computational capabilities of neural models raises the question of *hypercomputation*.
- Current physical theories are consistent with the possibility of hypercomputational systems (quantum, relativistic, etc.). No such systems are currently feasible or harnessable.
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